

CALCULUS II REVISION

IDENTITIES

We state the ff without proof.

Let k be any positive integer and c a constant.

Then the ff hold

$$\textcircled{1} \sum_{i=1}^k c = kc$$

$$\textcircled{3} \sum_{i=1}^k i^2 = \frac{k(k+1)(2k+1)}{6}$$

$$\textcircled{2} \sum_{i=1}^k i = \frac{k(k+1)}{2}$$

$$\textcircled{4} \sum_{i=1}^k i^3 = \left(\frac{k(k+1)}{2} \right)^2$$

$$\textcircled{5} \sum_{i=1}^k (ca_i + db_i) = c \sum_{i=1}^k a_i + d \sum_{i=1}^k b_i$$

Example

Solve $\sum_{n=1}^{100} n^2 + 2n - 4$

Soln

$$\begin{aligned} \sum_{n=1}^{100} n^2 + 2n - 4 &= \sum_{n=1}^{100} n^2 + 2 \sum_{n=1}^{100} n - \sum_{n=1}^{100} 4 \\ &= \frac{100(100+1)(2 \times 100 + 1)}{6} + 2 \left(\frac{100(100+1)}{2} \right) - 4 \times 100 \\ &= 348050 \end{aligned}$$

Riemann Sum

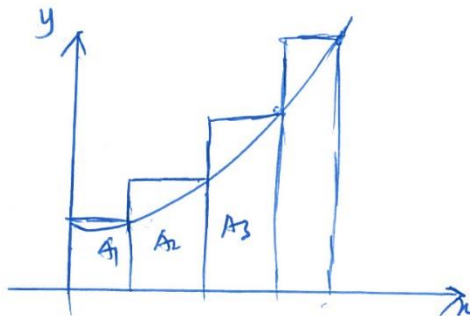
To find the area bounded by a curve.

Notation

$$\text{width of rectangle} = \Delta x = \frac{b-a}{n}$$

$$\text{Height of rectangle} = f(x)$$

$$\text{number of rectangles} = n$$



$$\text{Total Area} = A_1 + A_2 + A_3 + \dots + A_n$$

$$= \Delta x f(x_1) + f(x_2) \Delta x + \Delta x f(x_3) + \dots + \Delta x f(x_n)$$

$$= \Delta x (f(x_1) + f(x_2) + f(x_3) + \dots + f(x_n))$$

$$A = \sum_{i=1}^n \Delta x f(x_i) \longrightarrow \text{Right Riemann Sum.}$$

Riemann Sum as n becomes larger

$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

$$\text{where } x_i = x_0 + i \Delta x$$

$$\text{and } x_0 = a$$

Definite Integral

Given a function $f(x)$ that is continuous on the interval $[a, b]$ we divide the interval into n subintervals of equal width, Δx , and from each interval choose a point, x_i . The definite integral of $f(x)$ from a to b is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

Example

Evaluate $\int_0^1 x^2 + 1 dx$

Soln
 $f(x) = x^2 + 1$

$$\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

$$\textcircled{1} \Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$\begin{aligned} \textcircled{2} x_i &= x_0 + i\Delta x \\ &= 0 + i\left(\frac{1}{n}\right) \\ &= \frac{i}{n} \end{aligned}$$

$$\begin{aligned} \textcircled{3} f(x_i) &= (x_i)^2 + 1 \\ &= \left(\frac{i}{n}\right)^2 + 1 \\ &= \frac{i^2}{n^2} + 1 \end{aligned}$$

$$\begin{aligned} \textcircled{4} f(x_i) \Delta x &= \left(\frac{i^2}{n^2} + 1\right) \left(\frac{1}{n}\right) \\ &= \frac{i^2}{n^3} + \frac{1}{n} \end{aligned}$$

$$\begin{aligned} \textcircled{5} \sum_{i=1}^n f(x_i) \Delta x &= \sum_{i=1}^n \left(\frac{i^2}{n^3} + \frac{1}{n}\right) \\ &= \frac{1}{n^3} \sum_{i=1}^n i^2 + \frac{1}{n} \sum_{i=1}^n 1 \\ &= \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{n}(n) \\ &= \frac{1}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 1 \end{aligned}$$

$$\begin{aligned} \textcircled{6} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x &= \lim_{n \rightarrow \infty} \left[\frac{1}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 1 \right] \\ &= \frac{1}{6}(2) + 1 \\ &= \frac{4}{3} \\ \therefore \int_0^1 x^2 + 1 dx &= \frac{4}{3} \end{aligned}$$

Antiderivative / Indefinite Integral

Given a function, $f(x)$, an anti-derivative of $f(x)$ is any function $F(x)$

such that $F'(x) = f(x)$

If $F(x)$ is any anti-derivative of $f(x)$ then the most general anti-derivative of $f(x)$ is called an indefinite integral and denoted,

$$\int f(x) dx = F(x) + C$$

$\downarrow$ Integrand $\downarrow$ Anti-derivative

— Constant of integration

Example

$$\int x^4 + 3x dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 + C$$

Note the ff

$$\textcircled{1} \int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\textcircled{2} \int k dx = kx + C \quad \text{where } k \text{ and } C \text{ are constants.}$$

$$\textcircled{3} \int e^x dx = e^x + C$$

$$\textcircled{4} \int a^x dx = \frac{a^x}{\ln(a)} + C$$

$$\textcircled{5} \int \frac{1}{x} dx = \int x^{-1} dx = \ln|x| + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \operatorname{cosec}^2 x \, dx = -\cot x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$$

$$\int \frac{1}{x^2+1} \, dx = \tan^{-1}(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}(x) + C$$

or

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = -\cos^{-1}(x) + C$$

Example

$$\int 2^{x+7} \, dx = \frac{2^{x+7}}{\ln 2} + C$$

$$\int 5^{3x+8} \, dx = \frac{5^{3x+8}}{3 \ln 5} + C$$

$$\begin{aligned} \int \frac{e^x+1}{e^x} \, dx &= \int 1 + e^{-x} \, dx \\ &= x - e^{-x} + C \end{aligned}$$

$$\int \sin 2x \, dx = -\frac{\cos 2x}{2} + C$$

$$\int \sin(\pi+2x) \, dx = -\frac{\cos(\pi+2x)}{2} + C$$

$$\int \sin^2 x \, dx = \int \frac{1}{2} (1 - \cos 2x) \, dx$$

$$= \frac{1}{2}x - \frac{\sin 2x}{4} + C$$

NB: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Substitution

Eg. solve.

$$\int 2x(x^2+1)^3 dx$$

let $u = x^2 + 1$

$$\frac{du}{dx} = 2x$$

$$du = 2x dx$$

$$\int 2x(x^2+1)^3 dx = \int u^3 du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{(x^2+1)^4}{4} + C$$

$$\int e^t + e^{et} dt$$

$$= \int e^t \cdot e^{et} dt$$

let $u = e^t$

$$\frac{du}{dt} = e^t$$

$$du = e^t dt$$

$$\Rightarrow \int e^t + e^{et} dt = \int e^u du$$

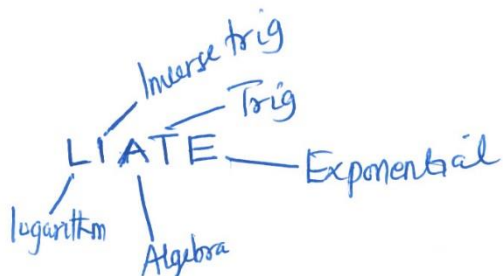
$$= e^u + C$$

$$= e^{et} + C$$

Integration By Parts

$$\int u dv = uv - \int v du$$

Order of choosing "u" is



Example

① $\int x e^x dx$

let $u = x$ $dv = e^x dx$

$\frac{du}{dx} = 1$ $v = e^x$

$du = dx$

$$\int x e^x dx = x e^x - \int e^x dx$$

$$= \underline{x e^x - e^x + C}$$

② $\int x^2 \cos x dx$

$u = x^2$ $dv = \cos x dx$
 $du = 2x dx$ $v = \sin x$

$$\int x^2 \cos x dx = x^2 \sin x - \int \sin x \cdot 2x dx$$

$$= x^2 \sin x - 2 \int x \sin x dx$$

$\int x \sin x dx$

$u = x$ $dv = \sin x dx$

$du = dx$ $v = -\cos x$

$$\int x \sin x dx = -x \cos x - \int -\cos x dx$$

$$= -x \cos x + \int \cos x dx$$

$$= -x \cos x + \sin x + C$$

$\Rightarrow \int x^2 \cos x dx$

$$= x^2 \sin x (-2x \cos x + 2 \sin x) + C$$

$$= \underline{x^2 \sin x + 2x \cos x - 2 \sin x + C}$$

Alternative

+	x^2	$\cos x$
-	$2x$	$\sin x$
+	2	$-\cos x$
-	0	$-\sin x$

$\Rightarrow \int x^2 \cos x dx$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + C$$

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Trigonometric Substitution

$$\begin{aligned} &\sqrt{a^2 - b^2 x^2} \\ \sin \theta &= \sqrt{1 - \cos^2 \theta} \\ \cos \theta &= \sqrt{1 - \sin^2 \theta} \\ \Rightarrow x &= \frac{a}{b} \sin \theta \end{aligned}$$

$$\begin{aligned} &\sqrt{b^2 x^2 - a^2} \\ \tan \theta &= \sqrt{\sec^2 \theta - 1} \\ \sec \theta &= \sqrt{1 + \tan^2 \theta} \\ x &= \frac{a}{b} \sec \theta \end{aligned}$$

$$\begin{aligned} &\sqrt{a^2 + b^2 x^2} \\ \sec \theta &= \sqrt{1 + \tan^2 \theta} \\ x &= \frac{a}{b} \tan \theta \end{aligned}$$

Example

$$\int \sqrt{25 - x^2} \, dx$$

Soln

$$\begin{aligned} \int \sqrt{25 - x^2} \, dx &= \int \sqrt{25 \left(1 - \frac{x^2}{25}\right)} \, dx \\ &= \int 5 \sqrt{1 - \left(\frac{x}{5}\right)^2} \, dx \end{aligned}$$

$$\frac{x}{5} = \sin \theta$$

$$x = 5 \sin \theta$$

$$dx = 5 \cos \theta \, d\theta$$

$$\Rightarrow \int 5 \sqrt{1 - \left(\frac{x}{5}\right)^2} \, dx = 5 \int \sqrt{1 - \sin^2 \theta} \cdot 5 \cos \theta \, d\theta$$

$$= 25 \int \cos \theta \cdot \cos \theta \, d\theta$$

$$= 25 \int \cos^2 \theta \, d\theta$$

NB: $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$

$$\begin{aligned} \Rightarrow \int \sqrt{25 - x^2} \, dx &= 25 \int \frac{1}{2}(1 + \cos 2\theta) \, d\theta \\ &= \frac{25}{2} \theta + \frac{25}{4} \sin 2\theta + \end{aligned}$$

$$\theta = \sin^{-1} \left(\frac{x}{5} \right)$$

$$\int \sqrt{25 - x^2} \, dx = \frac{25}{2} \sin^{-1} \left(\frac{x}{5} \right) + \frac{25}{4} \sin^2 \left(\sin^{-1} \left(\frac{x}{5} \right) \right)$$

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Integrals Involving Trig Functions

$$\begin{aligned}\textcircled{1} \int \cos^4 x \, dx &= \int (\cos^2 x)^2 \, dx \\&= \int \left[\frac{1}{2}(1 + \cos 2x) \right]^2 \, dx \\&= \frac{1}{4} \int 1 + 2\cos 2x + \cos^2 2x \, dx \\&= \frac{1}{4} \int 1 + 2\cos 2x + \left(\frac{1}{2} + \frac{\cos 4x}{2} \right) \, dx \\&= \frac{1}{4} \int \frac{3}{2} + 2\cos 2x + \frac{\cos 4x}{4} \, dx \\&= \frac{1}{8} \left(3x + 2\sin 2x + \frac{\sin 4x}{4} \right) + C\end{aligned}$$

$$\begin{aligned}\textcircled{2} \int \sin^5 x \, dx &= \int \sin^4 x \cdot \sin x \, dx \\&= \int (\sin^2 x)^2 \cdot \sin x \, dx \\&= \int (1 - \cos^2 x)^2 \cdot \sin x \, dx \\&\quad \text{let } u = \cos x \\&\quad \quad -du = \sin x \, dx\end{aligned}$$

$$\begin{aligned}\Rightarrow \int \sin^5 x \, dx &= \int (1 - u^2)^2 \cdot -du \\&= - \int 1 - 2u^2 + u^4 \, du \\&= - \left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5 \right) + C \\&= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C\end{aligned}$$

$$\textcircled{3} \int \cos x \sin^5 x \, dx$$

$$\text{let } u = \sin x \\ du = \cos x \, dx$$

$$\begin{aligned} \Rightarrow \int \cos x \sin^5 x \, dx &= \int u^5 \, du \\ &= \frac{u^6}{6} + C \\ &= \frac{1}{6} \sin^6 x + C \end{aligned}$$

$$\textcircled{4} \int \sin^6 x \cos^3 x \, dx$$

$$= \int \sin^6 x \cos^2 x \cos x \, dx$$

$$= \int \sin^6 x (1 - \sin^2 x) \cos x \, dx$$

$$\text{let } u = \sin x \\ du = \cos x \, dx$$

$$\begin{aligned} \Rightarrow \int \sin^6 x \cos^3 x \, dx &= \int u^6 (1 - u^2) \, du \\ &= \int u^6 - u^8 \, du \\ &= \frac{u^7}{7} - \frac{u^9}{9} + C \\ &= \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C \end{aligned}$$

Summary of integrating powers of sine and cosine

$$\int \sin^n x \cos^m x \, dx$$

In this integral if the exponent on the sine (n) is odd we can strip out one sine, convert the rest to cosines using $\sin^2 x + \cos^2 x = 1$ and then use the substitution $u = \cos x$. Likewise, if the exponent on the cosine (m) is odd we can strip out one cosine and convert the rest to sines and use the substitution $u = \sin x$.

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If both exponents are odd then we can use ~~either~~ either method. However, in these cases it's usually easier to convert the term with the smaller exponent

Example: Evaluate $\int \sin^5 x \cos^3 x dx$

$$= \int \sin^5 x \cos^2 x \cdot \cos x dx$$

$$= \int \sin^5 x (1 - \sin^2 x) \cos x dx$$

$$\text{let } u = \sin x$$

$$du = \cos x dx$$

$$= \int u^5 (1 - u^2) du$$

$$= \int u^5 - u^7 du$$

$$= \frac{u^6}{6} - \frac{u^8}{8} + C$$

$$= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C$$

Note the ff

$$\sin x \cos x = \frac{1}{2} \sin(2x)$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

Try: Evaluate: $\int \cos^2 x \sin^2 x dx$

Evaluate $\int \cos(15x) \cos(4x) dx$

$$= \frac{1}{2} \int \cos(15x-4x) + \cos(15x+4x) dx$$

$$= \frac{1}{2} \int \cos(11x) + \cos(19x) dx$$

$$= \frac{1}{2} \left(\frac{\sin(11x)}{11} + \frac{\sin(19x)}{19} \right) + C$$

Evaluation of $\int \sec^n x \tan^m x dx$ use: $\tan^2 x + 1 = \sec^2 x$

o Example: $\int \sec^9 x \tan^5 x dx$

$$= \int \sec^8 x \tan^4 x \sec x \tan x dx$$

$$= \int \sec^8 x (\sec^2 x - 1) \sec x \tan x dx$$

$$\text{let } u = \sec x \\ du = \sec x \tan x dx$$

$$= \int u^8 (u^2 - 1) du$$

$$= \int u^{12} - 2u^{10} + u^8 du$$

$$= \frac{u^{13}}{13} - \frac{2u^{11}}{11} + \frac{u^9}{9} + C$$

$$= \frac{1}{13} \sec^{13} x - \frac{2}{11} \sec^{11} x + \frac{1}{9} \sec^9 x + C$$

$$\begin{aligned}
 & \textcircled{2} \int \sec^4 x \tan^6 x \, dx \\
 &= \int \sec^2 x \tan^6 x \sec^2 x \, dx \\
 &= \int (\tan^2 x + 1) \tan^6 x \sec^2 x \, dx \\
 &\quad \text{let } u = \tan x \\
 &\quad du = \sec^2 x \, dx
 \end{aligned}$$

$$\begin{aligned}
 &= \int (u^2 + 1) u^6 \, du \\
 &= \int u^8 + u^6 \, du \\
 &= \frac{1}{9} \tan^9 x + \frac{1}{7} \tan^7 x + C
 \end{aligned}$$

Summary

We can easily convert even powers of secants to tangents and even powers of tangents to secants by using the formula $\tan^2 x + 1 = \sec^2 x$.

① If we use the substitution $u = \tan x$ we will need two secants left for the substitution to work. This means that if the exponent on the secant is even we can strip two out and then convert the remaining to tangents.

② If we want to use the substitution $u = \sec x$ we will need one secant and one tangent left over in order to use the substitution. This means that if the exponent on the tangent is odd and we have at least one secant in the integrand we can strip out one of the tangents along with one of the secants of course.

~~③ If the~~

Partial Fractions

NB: $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$

eg. $\int \frac{2x-1}{x^2-x+6} dx = \ln|x^2-x+6| + C$

① Solve $\int \frac{3x+11}{x^2-x-6} dx$

$$\frac{3x+11}{x^2-x-6} = \frac{4}{x-3} - \frac{1}{x+2}$$

$$\begin{aligned} \Rightarrow \int \frac{3x+11}{x^2-x-6} dx &= \int \frac{4}{x-3} - \frac{1}{x+2} dx \\ &= 4\ln|x-3| - \ln|x+2| + C \end{aligned}$$

② Solve $\int \frac{x^2+4}{3x^3+4x^2-4x} dx$

$$\begin{aligned} \frac{x^2+4}{3x^3+4x^2-4x} &= \frac{x^2+4}{x(x+2)(3x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{3x-2} \\ &= -\frac{1}{x} + \frac{\frac{1}{2}}{x+2} + \frac{\frac{5}{2}}{3x-2} \end{aligned}$$

$$\begin{aligned} \Rightarrow \int \frac{x^2+4}{3x^3+4x^2-4x} dx &= \int -\frac{1}{x} + \frac{\frac{1}{2}}{x+2} + \frac{\frac{5}{2}}{3x-2} dx \\ &= -\ln|x| + \frac{1}{2}\ln|x+2| + \frac{5}{6}\ln|3x-2| + C \end{aligned}$$

Improper Integrals

- ✓ Infinite interval
- ✓ Discontinuous integrand

Examples

① Evaluate $\int_1^{\infty} x \, dx$

This improper because the interval is infinite (not closed)

$$\begin{aligned}\int_1^{\infty} x \, dx &= \int_1^t x \, dx \\ &= \left. \frac{1}{2}x^2 \right|_1^t \\ &= \frac{t^2}{2} - \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\lim_{t \rightarrow \infty} \int_1^t x \, dx &= \lim_{t \rightarrow \infty} \frac{t^2}{2} - \frac{1}{2} \\ &= \infty\end{aligned}$$

$$\therefore \int_1^{\infty} x \, dx = \infty$$

$\therefore$ the integral diverges

$$\textcircled{2} \int_1^{\infty} \frac{1}{x^2} dx$$

It's improper becoz its has an infinite interval.

$$\int_1^{\infty} \frac{1}{x^2} dx = \int_1^t \frac{1}{x^2} dx$$

$$= -\frac{1}{x} \Big|_1^t$$

$$= 1 - \frac{1}{t}$$

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} dx$$

$$= \lim_{t \rightarrow \infty} \left(1 - \frac{1}{t}\right)$$

$$\therefore \int_1^{\infty} \frac{1}{x^2} dx = 1$$

It converges.

$$\textcircled{3} \text{ Evaluate } \int_1^0 \frac{1}{x^2} dx$$

It's improper becoz the integrand is discontinuous at $x=0$.

$$\text{c) } \int_1^0 \frac{1}{x^2} dx = \int_1^t \frac{1}{x^2} dx$$

$$= 1 - \frac{1}{t}$$

$$\int_1^0 \frac{1}{x^2} dx = \lim_{t \rightarrow 0^+} \int_1^t \frac{1}{x^2} dx$$

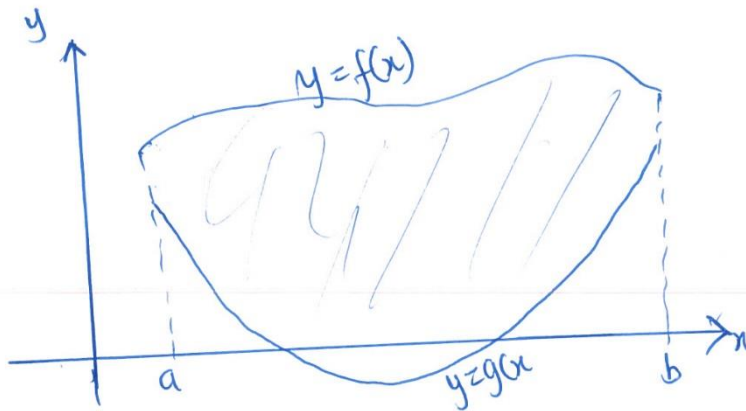
$$= \lim_{t \rightarrow 0^+} \left(1 - \frac{1}{t}\right)$$

$$= 1 - \infty$$

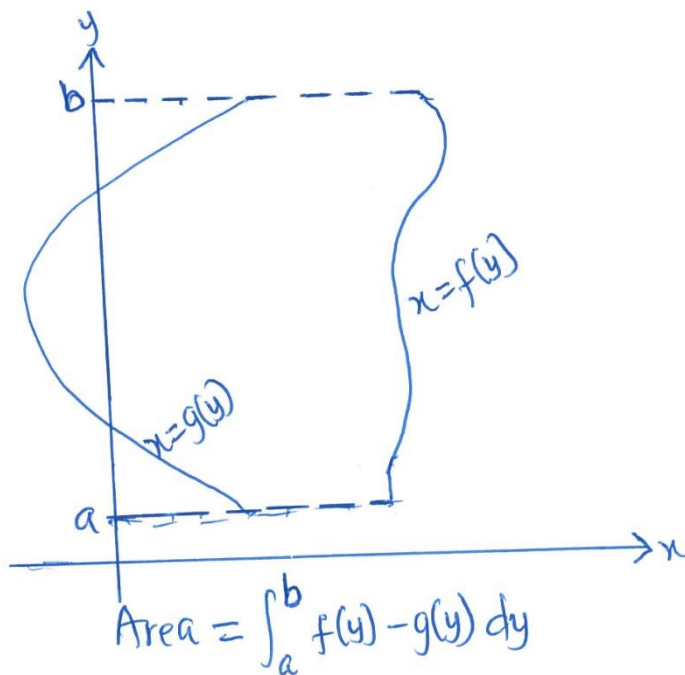
$$= -\infty$$

It diverges

Area b/n Curves



$$\text{Area} = \int_a^b f(x) - g(x) dx$$



$$\text{Area} = \int_a^b f(y) - g(y) dy$$

Average Function Value

The average value of a function $f(x)$ over the interval $[a, b]$ is given by,

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$$

Volume of Solids of Revolution

$$V = \int_a^b A(x) dx$$

- $A(x)$ is the cross-sectional area of the solid.

$$V = \pi \int_a^b (f(x))^2 dx$$

Probability Density Function (PDF.)

Satisfy the ff conditions

1. $f(x) \geq 0$ (ie graph lies above the x-axis)
2. $\int_a^b f(x) dx = 1$ (Total area under the curve is 1).

The probability is denoted by $P(a \leq x \leq b)$ and
is given by $P(a \leq x \leq b) = \int_a^b f(x) dx$

The mean ^{or expected value} is given by:

$$E(x) = \int_a^b x f(x) dx$$

Median is given by $\int_a^T f(x) dx = \frac{1}{2}$ or $\int_T^b f(x) dx = \frac{1}{2}$

where T is the median b/n a and b .

1. If $f(x) = \frac{1}{2}$ on $[1, 3]$ is a pdf, find the ff

1. $P(2 \leq x \leq 3)$
2. $P(x=1)$
3. Mean
4. Median

Soln

$$\begin{aligned} (1) P(2 \leq x \leq 3) &= \int_2^3 \frac{1}{2} dx \\ &= \frac{1}{2} x \Big|_2^3 \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} (2) P(x=1) &= \int_1^1 \frac{1}{2} dx \\ &= 0 \end{aligned}$$

$$\begin{aligned} (3) \text{ Mean} &= \int_1^3 x f(x) dx \\ &= \int_1^3 x \left(\frac{1}{2}\right) dx \\ &= \frac{x^2}{4} \Big|_1^3 \\ &= \underline{\underline{2.0}} \end{aligned}$$

$$\begin{aligned} (4) \text{ Median, } T & \\ \int_1^T \frac{1}{2} dx &= \frac{1}{2} \\ \frac{1}{2} x \Big|_1^T &= \frac{1}{2} \\ \frac{T}{2} - \frac{1}{2} &= \frac{1}{2} \\ \frac{T}{2} &= 1 \\ T &= 2 \end{aligned}$$

$\therefore$ the median is 2.

Differential Equations (DE)

Equations containing derivatives

Order: Highest derivative present.

1st Order Linear DE

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Example

Find the solution to $t \frac{dy}{dt} + 2y = t^2 - t + 1$

Soln

$$t \frac{dy}{dt} + 2y = t^2 - t + 1$$

$$\frac{dy}{dt} + \frac{2y}{t} = t - 1 + \frac{1}{t}$$

$$\begin{aligned} \text{Integration Factor (I.F.)} &= e^{\int \frac{2}{t} dt} \\ &= e^{2 \ln |t|} \\ &= t^2 \end{aligned}$$

Multiply Through by I.F

$$\Rightarrow t^2 \frac{dy}{dt} + t^2 \cdot \frac{2y}{t} = t^2 \left(t - 1 + \frac{1}{t} \right)$$

$$(t^2 y)' = t^3 - t^2 + t$$

$$\int (t^2 y)' dt = \int t^3 - t^2 + t dt$$

$$t^2 y = \frac{1}{4} t^4 - \frac{1}{3} t^3 + \frac{1}{2} t^2 + C$$

$$\Rightarrow y = \frac{1}{4} t^2 - \frac{1}{3} t + \frac{1}{2} + \frac{C}{t^2}$$

Separable Differential Equations

Form: $N(y) \frac{dy}{dx} = M(x)$

Re-write as: ~~dx~~ $N(y) dy = M(x) dx$

Integrate! $\int N(y) dy = \int M(x) dx$

Example

Solve: $\frac{dy}{dx} = 6y^2x$, $y(1) = \frac{1}{25}$

Soln

$$\frac{1}{y^2} \frac{dy}{dx} = 6x$$

$$y^{-2} dy = 6x dx$$

$$\int y^{-2} dy = \int 6x dx$$

$$-\frac{1}{y} = 3x^2 + C$$

Solve for C

$$-\frac{1}{\frac{1}{25}} = 3(1)^2 + C$$

$$C = -28$$

$$\Rightarrow -\frac{1}{y} = 3x^2 - 28$$

$$\Rightarrow y = \frac{1}{28 - 3x^2}$$

Series Representation of Functions

Taylor's Method

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots \end{aligned}$$

Maclaurin's Method

Taylor's method where $a=0$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots \end{aligned}$$

Taylor's Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Fourier Series

NB : Fact

① If $f(x)$ is an even function then,

$$\int_{-L}^L f(x) dx = 2 \int_0^L f(x) dx$$

② If $f(x)$ is an odd function then,

$$\int_{-L}^L f(x) dx = 0$$

Fourier series for $f(x)$ on $-L \leq x \leq L$

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Revise:

- ① Sequence and series
- ② Application of integrals
- ③ Parametric equations
- ④ Application of sequence and series