

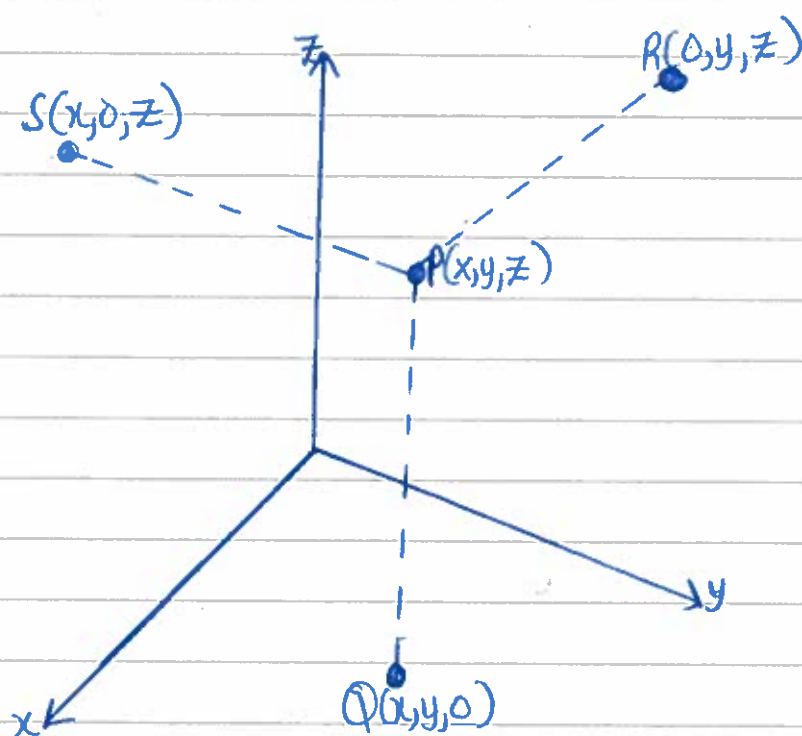
MULTIVARIABLE CALCULUS

The 3D Coordinate System

The 3-D coordinate system is often denoted by $\mathbb{R}^3$. Likewise the 2D coordinate system is often denoted by $\mathbb{R}^2$ and 1-D is $\mathbb{R}$.

It implies that the general 'n' dimensional coordinate system is often denoted by $\mathbb{R}^n$.

Look at the basic coordinate system.



Note that only the positive ~~axe~~ directions are shown by the axes. If we need the negative axes for any reason we will put them in as needed.

The point $P(x, y, z)$ is the general point sitting out in 3-D space.

- If we start at P and drop straight down until we reach a z -coordinate of zero we arrive at point $Q(x, y, 0)$. We say Q sits in the xy -plane. This implies that, the xy -plane corresponds to all points which have a zero z -coordinate.

- Start at P and move to point $S(x, 0, z) \rightarrow xz$ -plane.

- Start at P and move to point $R(0, y, z) \rightarrow yz$ -plane

- Collectively, the xy , xz and yz -planes are called the coordinate system.

✓ Point Q is often referred to as the projection of P in the xy -plane.

– R is the projection of P in the yz -plane.

– S is the projection of P in the xz -plane.

Distance b/n two points in $\mathbb{R}^2$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In $\mathbb{R}^3$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

General eqn for a circle with centre (a, b) and radius, r

$$(x - a)^2 + (y - b)^2 = r^2$$

General eqn for a sphere with centre (a, b, c) and radius, r

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

Question

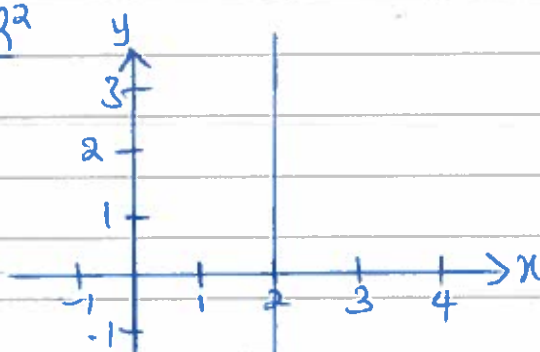
Graph $x = 2$ in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$

Solution

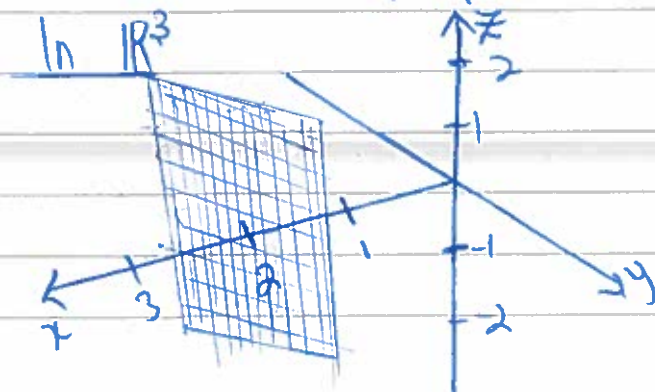
In $\mathbb{R}$



In $\mathbb{R}^2$



In $\mathbb{R}^3$



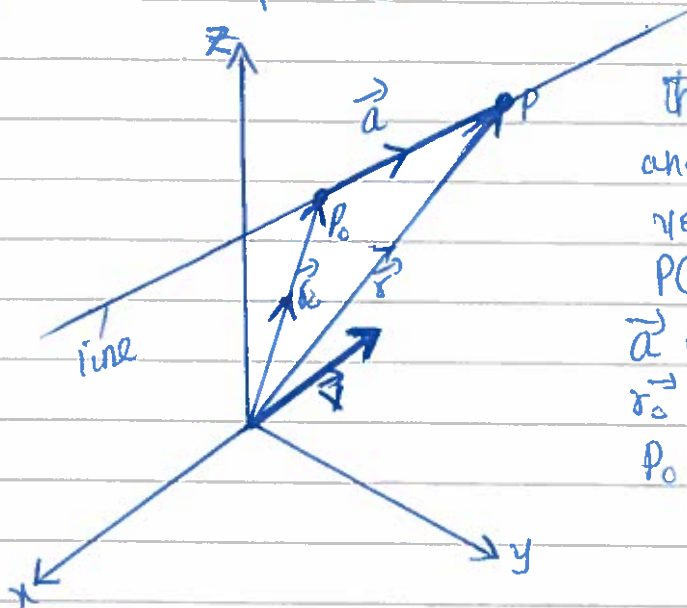
TRY

① Graph $y = x + 3$ in $\mathbb{R}^2$ and $\mathbb{R}^3$

② Graph $x^2 + y^2 = 4$ in $\mathbb{R}^2$ and $\mathbb{R}^3$

Equations of Lines

Note: The equation $y = mx + b$ does not describe a line in $\mathbb{R}^3$, instead it describes a plane.



Suppose that we know point that is on the line, $P_0(x_0, y_0, z_0)$ and that $\vec{v} = \langle a, b, c \rangle$ is some vector parallel to the line. Let $P(x, y, z)$ be any point on the line. $\vec{a}$ is the vector representation for $\vec{P_0P}$. $\vec{r_0}$ and $\vec{r}$ are position vectors for P_0 and P respectively.

From the diagram, $\vec{r} = \vec{r_0} + \vec{a}$

Notice that $\vec{a}$ and $\vec{v}$ are parallel. Therefore, there is a number, t , such that

$$\begin{aligned} \vec{a} &= t\vec{v} \\ \Rightarrow \vec{r} &= \vec{r_0} + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle \end{aligned}$$

↓
This is called the vector form of the equation of a line

We know, $\vec{r} = \langle x, y, z \rangle$

$$\Rightarrow \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

$$x = x_0 + ta$$

$$y = y_0 + tb$$

$$z = z_0 + tc$$

} Parametric form of the equation of a line

¶

If we assume that a, b and c are all non-zero numbers we can solve each of the equations in the parametric form of the line for t .

$$\Rightarrow \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

⏟
This is called the Symmetric Equations of the line.

Example 1

Write down the equation of the line that passes through the points $(2, -1, 3)$ and $(1, 4, -3)$. Write down all three forms of the equation of the line.

Solution

We first need the vector $\vec{v}$ that is parallel to the line.

$$\begin{aligned}\vec{v} &= \langle 2, -1, 3 \rangle - \langle 1, 4, -3 \rangle \\ &= \langle 1, -5, 6 \rangle\end{aligned}$$

Use any of point to find the equation.

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

$$\vec{r} = \langle 2, -1, 3 \rangle + t\langle 1, -5, 6 \rangle$$

$$\vec{r} = \langle 2+t, -1-5t, 3+6t \rangle \Rightarrow \text{Vector form of the line.}$$

for

Parametric form of the line

$$x = 2 + t$$

$$y = -1 - 5t$$

$$z = 3 + 6t$$

Symmetric form

$$\frac{x-2}{1} = \frac{y+1}{-5} = \frac{z-3}{6}$$

Example 2

Two lines are defined by:

$$L_1: \vec{r}_1(t) = \langle -2+3t, 2t, 3t \rangle$$

$$L_2: \vec{r}_2(s) = \langle -6+s, -8+2s, -12+3s \rangle$$

Are these lines,

① Parallel

② Intersecting. (If the point of intersection)

Solution

① Parallel.

Two lines are parallel if their direction vectors are equal or scalar multiples.

Extract the direction vectors: $\vec{v}_1 = \langle 3, 2, 3 \rangle$ $\vec{v}_2 = \langle 1, 2, 3 \rangle$

Since the direction vectors are not equal or scalar multiples, it implies that

① L_1 and L_2 are not parallel.

② Intersecting:

To check for point of intersection, equate them and find t and s . If t and s satisfy all three equations, then there's a point of intersection.

$$\vec{r}_1(t) = \vec{r}_2(s)$$

$$-2+3t = -6+s \quad \text{--- ①} \quad 2t = -8+2s \quad \text{--- ②} \quad 3t = -12+3s \quad \text{--- ③}$$

Solving the equations simultaneously gives

$$t=0 \quad \text{and} \quad s=4$$

$t=0$ and $s=4$ satisfy all three equations

$$\vec{r}_1(0) = \langle -2, 0, 0 \rangle$$

$$\vec{r}_2(0) = \langle -2, 0, 0 \rangle$$

$\therefore$ Their point of intersection is $(-2, 0, 0)$

TRY

$L_1: r(t) = \langle \cancel{2+2t}, 8+t, 10+3t \rangle$ ① Are they (a) Parallel (b) Intersecting

$L_2: R(s) = \langle 6+s, 10-2s, 16-s \rangle$ ② Find the point of intersection if there's any.

Review on Vectors in a Plane

Vector — Any quantity that has both magnitude and direction.

Equal Vector

Two vectors say $\vec{u}$ and $\vec{v}$ are equal if they have the same length and direction.

Scalar Multiplication, Vector Addition and Subtraction, Magnitude

Question Suppose c is a scalar, $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \quad \text{— addition}$$

$$\vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2 \rangle \quad \text{— subtraction}$$

$$c\vec{u} = \langle cu_1, cu_2 \rangle \quad \text{— scalar multiplication.}$$

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2} \quad \text{— magnitude}$$

Question

Let $\vec{a} = \langle 1, 2 \rangle$ and $\vec{b} = \langle 3, 4 \rangle$, Find

(a) $|\vec{a} + \vec{b}|$

(b) $2\vec{a} - 3\vec{b}$

(c) two vectors half as long as $\vec{a}$ and parallel to $\vec{a}$.

Solution

(a) $\vec{a} + \vec{b} = \langle 1+3, 2+4 \rangle = \langle 4, 6 \rangle$

$$|\vec{a} + \vec{b}| = \sqrt{4^2 + 6^2} = 2\sqrt{13}$$

(b) $2\vec{a} - 3\vec{b} = 2\langle 1, 2 \rangle - 3\langle 3, 4 \rangle$

$$= \langle 2, 4 \rangle - \langle 9, 12 \rangle$$

$$= \langle -7, -8 \rangle$$

(c) The vectors are $\frac{1}{2}\vec{a}$ and $-\frac{1}{2}\vec{a}$

$$-\frac{1}{2}\langle 1, 2 \rangle = \langle -\frac{1}{2}, -1 \rangle \quad \frac{1}{2}\langle 1, 2 \rangle = \langle \frac{1}{2}, 1 \rangle$$

$\therefore$ the vectors are $\langle -\frac{1}{2}, -1 \rangle$ and $\langle \frac{1}{2}, 1 \rangle$

Unit Vectors

A unit vector is any vector with length 1.

Coordinate unit vectors: $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$

Say we have a vector $\vec{u}$, the unit vector is given by $\frac{\vec{u}}{|\vec{u}|}$

Question

Say we have, $A(1, -2)$ and $B(6, 10)$. Find,

a. $\vec{AB}$ and two unit vectors parallel to $\vec{AB}$

b. two vectors of length 2 parallel to $\vec{AB}$.

Soln

$$a. \vec{AB} = \vec{OB} - \vec{OA}$$

$$= \langle 6, 10 \rangle - \langle 1, -2 \rangle$$

$$= \langle 5, 12 \rangle \text{ or } 5\mathbf{i} + 12\mathbf{j}$$

$$|\vec{AB}| = \sqrt{5^2 + 12^2} = 13$$

$$\text{A unit vector along } \vec{AB} \text{ is } \frac{\vec{AB}}{|\vec{AB}|} = \frac{1}{13} \langle 5, 12 \rangle$$

$$= \frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}$$

The unit vector parallel to $\vec{AB}$ and with opposite direction is $-\frac{5}{13}\mathbf{i} - \frac{12}{13}\mathbf{j}$

b. multiply the unit vector by ± 2

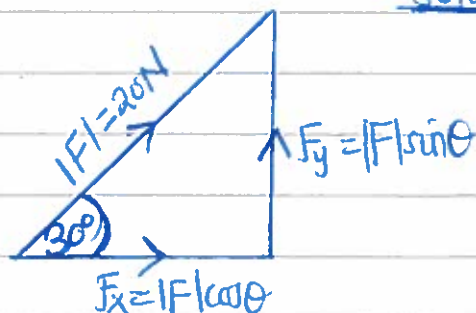
$$= \pm 2 \left(\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j} \right)$$

$$= \frac{10}{13}\mathbf{i} + \frac{24}{13}\mathbf{j} \text{ and } -\frac{10}{13}\mathbf{i} - \frac{24}{13}\mathbf{j}$$

Question

~~A child~~ Kwame Atta-Gyamfi pulls a wagon with a force of $|F| = 20\text{N}$ at an angle of $\theta = 30^\circ$ to the horizontal. Find the force vector, F .

Solution



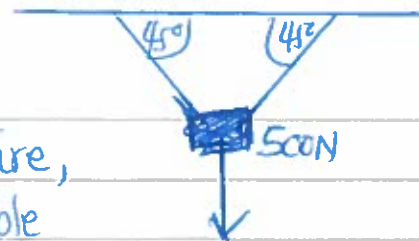
$$F = \langle |F|\cos\theta, |F|\sin\theta \rangle$$

$$= \langle 20\cos 30^\circ, 20\sin 30^\circ \rangle$$

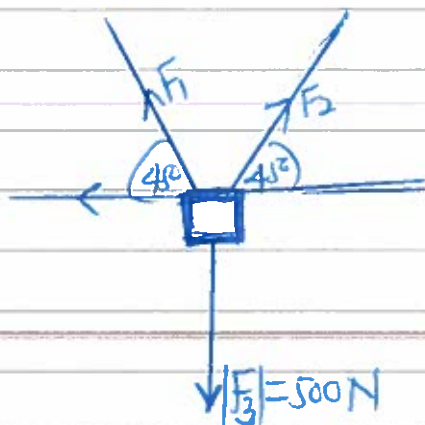
$$= \langle 10\sqrt{3}, 10 \rangle$$

Question

If a 500N load is suspended by two chains in the figure, what is the magnitude of force each chain must be able to support?



Solution



$$|F_1| = |F_2| = |F|$$

$$\begin{aligned} F_1 &= |F| \langle -\cos 45^\circ, \sin 45^\circ \rangle \\ &= \left\langle -\frac{F}{\sqrt{2}}, \frac{F}{\sqrt{2}} \right\rangle \end{aligned}$$

$$\begin{aligned} F_2 &= |F| \langle \cos 45^\circ, \sin 45^\circ \rangle \\ &= \left\langle \frac{F}{\sqrt{2}}, \frac{F}{\sqrt{2}} \right\rangle \end{aligned}$$

$$\begin{aligned} F_3 &= 500 \langle 0, -1 \rangle \\ &= \langle 0, -500 \rangle \end{aligned}$$

$$\begin{aligned} F_1 + F_2 + F_3 &= 0 \\ \left\langle -\frac{F}{\sqrt{2}}, \frac{F}{\sqrt{2}} \right\rangle + \left\langle \frac{F}{\sqrt{2}}, \frac{F}{\sqrt{2}} \right\rangle + \langle 0, -500 \rangle &= \langle 0, 0 \rangle \end{aligned}$$

$$\left\langle 0, \frac{2F}{\sqrt{2}} - 500 \right\rangle = \langle 0, 0 \rangle$$

$$\frac{2F}{\sqrt{2}} - 500 = 0$$

$$F = 250\sqrt{2} \text{ N}$$

∴ each chain must provide $250\sqrt{2} \text{ N}$ to be able to support.

Question

Find an inequality that describes a ball with centre $(0, -2, 6)$ with the point $(1, 4, 8)$ on its boundary.

Solution

$$\text{radius, } r = \sqrt{(0-1)^2 + (-2-4)^2 + (6-8)^2} \\ = \sqrt{41}$$

$$\text{Inequality: } (x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2$$

$$(a, b, c) = (0, -2, 6)$$

$$\Rightarrow (x-0)^2 + (y+2)^2 + (z-6)^2 \leq (\sqrt{41})^2 \\ \underline{x^2 + y^2 + z^2 + 4y - 12z \leq 1}$$

DOT PRODUCT

$$u \cdot v = |u||v|\cos\theta$$

* u and v are parallel ($\theta = 0$ or $\theta = \pi$) if and only $u \cdot v = \pm |u||v|$

* u and v are perpendicular ($\theta = \pi/2$) if $u \cdot v = 0$

Orthogonal Vectors

Two vectors u and v are orthogonal if and only if $u \cdot v = 0$

Question

Find the angle between ~~x and y~~ the vectors $x = \langle 1, 2, -3 \rangle$ and $y = \langle -2, 1, 1 \rangle$ to the nearest degree.

Solution.

$$x \cdot y = |x||y|\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{x \cdot y}{|x||y|}\right)$$

$$x \cdot y = (1 \times -2) + (2 \times 1) + (-3 \times 1) = -3$$

$$|x| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$$

$$|y| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}$$

$$\Rightarrow \text{angle, } \theta = \cos^{-1}\left(\frac{-3}{\sqrt{14} \times \sqrt{6}}\right)$$

$$= \underline{\underline{109^\circ}}$$

Question:

Are the points $A(-3, -2, 1)$, $B(1, 4, 7)$ and $C(4, 10, 14)$ collinear? Explain.

Solution

For the points to be collinear,

$$\vec{AC} = \lambda \vec{AB}$$

$$\begin{aligned}\vec{AB} &= \vec{OB} - \vec{OA} \\ &= \langle 4, 6, 6 \rangle\end{aligned}$$

$$\begin{aligned}\vec{AC} &= \vec{OC} - \vec{OA} \\ &= \langle 7, 12, 13 \rangle\end{aligned}$$

$$\vec{AC} = \lambda \vec{AB}$$

$$\langle 7, 12, 13 \rangle = \lambda \langle 4, 6, 6 \rangle$$

$$7 = 4\lambda$$

$$12 = 6\lambda$$

$$13 = 6\lambda$$

$$\lambda = \frac{7}{4}$$

$$\lambda = 2$$

$$\lambda = \frac{13}{6}$$

Since λ is not equal in all components, it implies that the points are not collinear.

Properties of Dot Product

$$(1) \quad u \cdot v = v \cdot u$$

$$(2) \quad c(u \cdot v) = (cu) \cdot v = u \cdot (cv)$$

$$(3) \quad u \cdot (v + w) = u \cdot v + u \cdot w$$

Questions

A force $F = \langle 3, 3, 2 \rangle$ (N) moves an object along a line segment $P(1, 1, 0)$ to $Q(6, 6, 0)$ (m). What is the work done by the force.

Solution

$$\text{Displacement, } d = \vec{PQ} = \vec{OQ} - \vec{OP} = \langle 5, 5, 0 \rangle$$

$$\text{Work done, } W = F \cdot d$$

$$= \langle 3, 3, 2 \rangle \cdot \langle 5, 5, 0 \rangle$$

$$= \underline{\underline{30 \text{ J}}}$$

Cross Product

let $u = u_1i + u_2j + u_3k$ and $v = v_1i + v_2j + v_3k$, Then

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

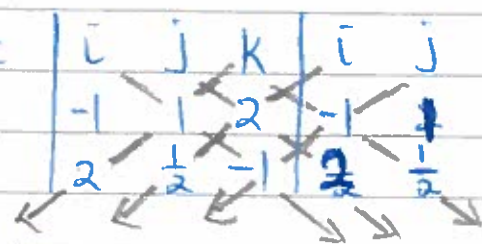
Question

If $\vec{u} = \langle -1, 1, 2 \rangle$ and $\vec{v} = \langle 2, \frac{1}{2}, -1 \rangle$ find $u \times v$

Soln

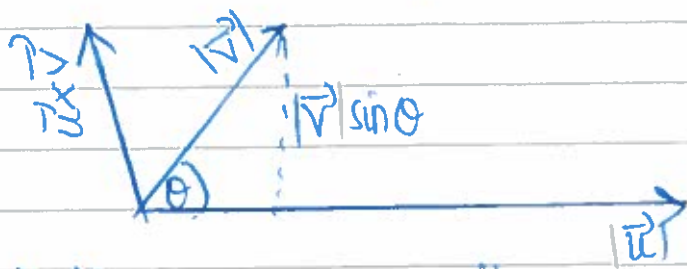
$$u \times v = \begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ 2 & \frac{1}{2} & -1 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 2 \\ \frac{1}{2} & -1 \end{vmatrix} i - \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} j + \begin{vmatrix} -1 & 1 \\ 2 & \frac{1}{2} \end{vmatrix} k$$
$$= -2i + 3j - \frac{5}{2}k$$

OR

$$u \times v = \begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ 2 & \frac{1}{2} & -1 \end{vmatrix}$$


$$u \times v = \left[(1 \times -1) + (2 \times 2) + (-1 \times \frac{1}{2}) \right] i - \left[(-1 \times -1) + (2 \times \frac{1}{2}) + (1 \times 2) \right] j + \left[(-1 \times \frac{1}{2}) + (1 \times 2) + (2 \times 2) \right] k$$
$$= -2i + 3j - \frac{5}{2}k$$

Area of a Parallelogram



$$|\vec{u} \times \vec{v}| = \text{area of parallelogram}$$

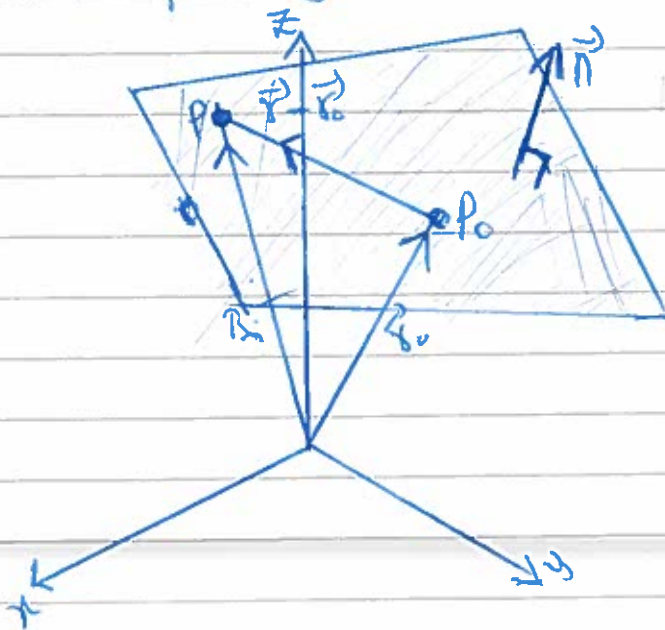
$$|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta$$

TRY! Find the area of the triangle with vertices $A(1,2,3)$, $B(5,1,5)$ and $C(2,3,3)$.

Hint! Area = $\frac{1}{2} |\vec{AB} \times \vec{AC}|$

EQUATIONS OF PLANES

Let's start by assuming that we know a point that is on the plane, $P_0 = (x_0, y_0, z_0)$. Suppose a vector that is orthogonal (perpendicular) to the plane, $\vec{n} = \langle a, b, c \rangle$. This vector is called the normal vector. Assume $P(x, y, z)$ is any point in the plane. Let $\vec{r}_0$ and $\vec{r}$ be the position vectors for P_0 and P respectively.



$\vec{n}$ is orthogonal to the plane and to any vector that lies in the plane.

$\vec{n}$ is orthogonal to $\vec{r} - \vec{r}_0$

$$\Rightarrow \vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

$$\Rightarrow \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \longrightarrow \text{Vector equation of the plane.}$$

$$\Rightarrow \langle a, b, c \rangle \cdot \langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle = 0$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Scalar equation of plane.

Often written as $ax + by + cz = d$.

Question

Determine the equation of the plane that contains the points $P(1, -2, 0)$, $Q(3, 1, 4)$ and $R(0, -1, 2)$

Solution

$$\vec{PQ} = \langle 2, 3, 4 \rangle \quad \vec{PR} = \langle -1, 1, 2 \rangle$$

normal vector, $\vec{n} = \vec{PQ} \times \vec{PR}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix}$$

$$= 2\mathbf{i} - 8\mathbf{j} + 5\mathbf{k}$$

$$\text{Equation: } a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$2(x - 1) - 8(y + 2) + 5(z - 0) = 0$$

$$\Rightarrow \underline{\underline{2x - 8y + 5z = 18}}$$

Question

Find the equation of the line through $(1, 2, 3)$ that is perpendicular to the lines $r_1(t) = \langle 3-2t, 5+8t, 7-4t \rangle$ and $r_2(t) = \langle -2t, 5+t, 7-t \rangle$

~~Answered~~

Solution

Directional vector for r_1 is $v_1 = \langle -2, 8, -4 \rangle$
for r_2 is $v_2 = \langle -2, 1, -1 \rangle$

$$\vec{n} = v_1 \times v_2$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 8 & -4 \\ -2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} \\ -2 & 8 \\ -2 & 1 \end{vmatrix}$$

$$= -4\hat{i} + 6\hat{j} + 14\hat{k}$$

Equation:

$$r(t) = p_0 + t(\vec{n})$$

$$= \langle 1, 2, 3 \rangle + t \langle -4, 6, 14 \rangle$$

$$= \langle 1-4t, 2+6t, 3+14t \rangle$$

MATRIX

A rectangular array of numbers.

Operations — Review

Determinant of 2×2

$$\text{if } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det(A) = a_{11}a_{22} - a_{21}a_{12}$$

~~det~~

Determinant of 3×3

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

① Suppose A is a square matrix

⑥ If $\det(A) = 0$ we call A a singular matrix.

⑥ If $\det(A) \neq 0$ we call A a non-singular matrix.

② Suppose that A is an $n \times n$ triangular matrix with diagonal entries $a_{11}, a_{22}, \dots, a_{nn}$ the determinant of A is

$$\det(A) = a_{11} a_{22} \dots a_{nn}$$

Example

Compute the det of the ff.

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 6 & 0 \\ 2 & -1 \end{pmatrix}$$

$$C = \begin{pmatrix} 10 & 5 & 1 & 3 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Solution

$$\det(A) = (5)(-3)(4) = -60$$

$$\det(B) = (6)(-1) = -6$$

$$\det(C) = (10)(0)(6)(5) = 0$$

③ If A is a square matrix then the minor of a_{ij} , denoted by M_{ij} , is the determinant of the submatrix that results from removing the i^{th} row and j^{th} column of A .

④ If A is a square matrix then the cofactor of a_{ij} , denoted by C_{ij} , is the number $(-1)^{i+j} M_{ij}$.

Question

Compute the cofactors C_{12} , C_{24} and C_{32}

$$A = \begin{pmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{pmatrix}$$

Solution

$$C_{12} = (-1)^{1+2} M_{12}$$

$$M_{12} = \begin{vmatrix} -1 & 3 & 9 \\ 5 & -1 & 6 \\ 3 & 1 & -2 \end{vmatrix} = 160$$

$$\therefore C_{12} = (-1)^{1+2} (160) = -160$$

$$C_{24} = (-1)^{2+4} M_{24}$$

$$M_{24} = \begin{vmatrix} 4 & 0 & 10 \\ 5 & -5 & -1 \\ 3 & 7 & 1 \end{vmatrix} = 508$$

$$C_{24} = (-1)^6 (508) = 508$$

$$C_{32} = (-1)^{3+2} M_{32}$$

$$M_{32} = \begin{vmatrix} 4 & 10 & 4 \\ -1 & 3 & 9 \\ 3 & 1 & -2 \end{vmatrix} = 150$$

$$C_{32} = (-1)^{3+2} (150) = -150$$

Properties of Determinant of a Matrix

- ① $|A| = |A^T|$
- ② Interchanging row/columns: If two rows/columns of A are interchanged, the determinant changes sign but the magnitude is the same.
- ③ Removing factor: If all the elements of a single row or column of A have a common factor, λ , then this factor may be removed. The value of the determinant is the product of the $|det|$ and λ .
- ④ If all the elements of any row or column is zero then the determinant is zero.
- ⑤ If any two rows/columns of A are identical or are multiples of one another then it can be shown that $|A| = 0$
- ⑥ Let A and B be $n \times n$ matrices
 $|AB| = |A||B|$

Questions

Let A and B be a 3×3 matrices with $\det(A) = 4$ and $\det(B) = 8$. Find

- (i) $\det(A^{-1}B)$
- (ii) $\det(3A)$
- (iii) $\det(AB)$
- (iv) $\det(B^{-1}A)$
- (v) $\det(2AB)$

Solution

- (i) $\det(A^{-1}B) = \det(A^{-1}) \times \det B$
 $= \frac{1}{\det(A)} \times \det B$
 $= \frac{1}{4} \times 8 = 2$
- (ii) $\det(3A) = \det A \times 3^3$
 $= 4 \times 3^3$
 $= 108$
- (iii) $\det(AB) = \det(A) \times \det(B)$
 $= 4 \times 8$
 $= 32$
- (iv) $\det(B^{-1}A) = \det(B^{-1}) \times \det A$
 ~~$= \frac{1}{\det B} \times \det A$~~
 $= \frac{1}{8} \times 4$
 $= \frac{1}{2}$
- (v) $\det(2AB) = \det(A) \times \det(B) \times 2^3$
 ~~$= \det(A)$~~
 $= 4 \times 8 \times 2^3$
 $= 256$

Inverse of a Matrix

$$A^{-1} = \frac{1}{\det(A)} \text{Adj}(A)$$

Adj $\rightarrow$ Adjoint

$$\text{Adj}(A) = C^T$$

C - Cofactor matrix

Eg.

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{pmatrix}$$

find A^{-1}

$$M_{11} = \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix}$$

$$M_{12} = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$$

$$M_{13} = \begin{bmatrix} 1 & -2 \\ -3 & 3 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

$$M_{22} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$$

$$M_{23} = \begin{bmatrix} 2 & 4 \\ -3 & 3 \end{bmatrix}$$

$$M_{31} = \begin{bmatrix} 4 & 3 \\ -2 & -2 \end{bmatrix}$$

$$M_{32} = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$$

$$M_{33} = \begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} |M_{11}|$$

$$C_{21} = (-1)^{2+1} |M_{21}|$$

$$C_{31} = (-1)^{3+1} |M_{31}|$$

$$C_{12} = (-1)^{1+2} |M_{12}|$$

$$C_{22} = (-1)^{2+2} |M_{22}|$$

$$C_{32} = (-1)^{3+2} |M_{32}|$$

$$C_{13} = (-1)^{1+3} |M_{13}|$$

$$C_{23} = (-1)^{2+3} |M_{23}|$$

$$C_{33} = (-1)^{3+3} |M_{33}|$$

$$\Rightarrow C = \begin{pmatrix} 2 & 4 & -3 \\ 1 & 13 & -18 \\ -2 & 7 & -8 \end{pmatrix}$$

$$\text{Adj}(A) = C^T = \begin{pmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{11} \begin{pmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{pmatrix}$$

Properties of Inverse Matrix

- ① $(A^{-1})^{-1} = A$
- ② $(A^T)^{-1} = (A^{-1})^T$
- ③ $(AB)^{-1} = B^{-1}A^{-1}$

The Rank of a Matrix

The rank r of a $n \times n$ matrix A is equal to the size of the largest square submatrix of A whose determinant is not zero.

Orthogonal Matrix

When $A^T = A^{-1}$

Diagonal Matrix

A matrix having non-zero elements only in the leading diagonal.

Symmetric Matrix

A square matrix which is equal to its transpose. $A = A^T$

Antisymmetric Matrix

$$A = -A^T$$

Try $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Trace of a Matrix

The trace of $n \times n$ matrix is the sum of the elements of the leading diagonal.

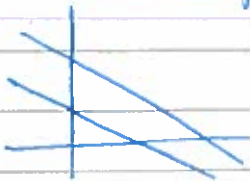
ie $\text{tr}(A) = \sum_{i=1}^n a_{11} + a_{22} + \dots + a_{nn}$

SYSTEMS OF EQUATIONS

- * When a system of linear equations have a solution, we say the system is consistent.
- * When the system has no solution, the system is inconsistent.
- * A system of eqns have a unique solution if they have a common point of intersection.

Example

① $x+y=2$
 $x+y=1$



If they have no solution, it implies they are parallel.

② $x+y=2$
 $x-y=2$ } $x=2$
 $y=0$ Unique solution

③ $x+y=2$
 $-x-y=-2$ } Both have the same line.
Infinite soln.

Try later: $2x+y+3z=1$
 $4x+3y+5z=1$
 $6x+5y+5z=-3$

Gaussian Elimination — Method 1

Pivot Position

✓ Only non-zero numbers

✓ Any number below the pivot should be zero.

Example.

$$2x + y + z = 1$$

$$6x + 2y + z = -1$$

$$-2x + 2y + z = 7$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 6 & 2 & 1 & -1 \\ -2 & 2 & 1 & 7 \end{array} \right) \begin{array}{l} \text{Soln} \\ \hline -R_1 \\ -R_2 \\ -R_3 \end{array}$$

Called the
Augmented Matrix

$$\begin{array}{l} 3R_1 - R_2 \rightarrow R_2 \\ R_1 + R_3 \rightarrow R_3 \end{array} \left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 2 & 8 \end{array} \right) \begin{array}{l} \\ \\ \end{array} \xrightarrow{3R_2 - R_3 \rightarrow R_3} \left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 4 & 4 \end{array} \right)$$

Back substitution,

$$4z = 4, \quad y + 2(1) = 4, \quad 2x + 2 + 1 = 1$$

$$z = 1$$

$$y = 2$$

$$x = -1$$

∴ Solution is $(-1, 2, 1)$

Row Echelon Form — Method 2

A matrix is said to be in row echelon form if;

- ① The first non-zero entry in each non-zero entry row is 1.
- ② If row k does not consist entirely of zeroes, the number of leading zero entries in row $k+1$ is greater than the no. of leading zero entries in row k .
- ③ If there are rows whose entries are all zero, they are below the rows having non-zero entries.

Examples: $\begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Gauss — Jordan Elimination — Method 3

The idea is to convert the matrix into an identity matrix. It implies that, it applies to only square matrices.

STEPS

- ✓ First is to make the first entry in the augmented matrix 1.
- ✓ Interchange Two Rows
- ✓ Multiply a Row by a Constant.
- ✓ Add a Multiple of a Row to another Row.

Example

Solve $2x - y = 0$
 $-x + 2y - z = 0$
 $0 - y + z = 0$

Solution

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array} \right)$$

$$\begin{aligned}
 & -R_1 \rightarrow R_1 \rightarrow \begin{pmatrix} 1 & -2 & 1 & | & 0 \\ 2 & -1 & 0 & | & 0 \\ 0 & -1 & 1 & | & 0 \end{pmatrix} \quad 2R_1 - R_2 \rightarrow R_2 \rightarrow \begin{pmatrix} 1 & -2 & 1 & | & 0 \\ 0 & -3 & 2 & | & 0 \\ 0 & -1 & 1 & | & 0 \end{pmatrix} \\
 & R_2 - 3R_3 \rightarrow R_2 \rightarrow \begin{pmatrix} 1 & -2 & 1 & | & 0 \\ 0 & -3 & 2 & | & 0 \\ 0 & 0 & -1 & | & 0 \end{pmatrix} \quad 2R_3 + R_2 \rightarrow R_2 \rightarrow \begin{pmatrix} 1 & -2 & 1 & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & -1 & | & 0 \end{pmatrix} \\
 & R_3 + R_1 \rightarrow R_1 \rightarrow \begin{pmatrix} 1 & -2 & 0 & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & -1 & | & 0 \end{pmatrix} \quad 3R_1 - 2R_3 \rightarrow R_1 \rightarrow \begin{pmatrix} 3 & 0 & 0 & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & -1 & | & 0 \end{pmatrix} \\
 & \frac{1}{3}R_1 \rightarrow R_1 \rightarrow \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & -1 & | & 0 \end{pmatrix} \\
 & -\frac{1}{3}R_2 \rightarrow R_2 \rightarrow \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & -1 & | & 0 \end{pmatrix} \\
 & -R_3 \rightarrow R_3 \rightarrow \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}
 \end{aligned}$$

$\therefore x=0, y=0 \text{ and } z=0$

Using Gauss-Jordan method to find the inverse of a matrix.

$$[A|I_n] \rightarrow [I_n|C]$$

$C \text{ is the } A^{-1}$

Example

Find the inverse of $A = \begin{pmatrix} 0 & 1 & -1 \\ -2 & 4 & -1 \\ -2 & 5 & -4 \end{pmatrix}$

Solution

$$\begin{aligned}
 & \equiv \left(\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ -2 & 4 & -1 & 0 & 1 & 0 \\ -2 & 5 & -4 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} r_2 \leftrightarrow r_1 \\ r_1 - r_3 \rightarrow r_3 \\ r_2 + r_3 \rightarrow r_3 \end{array} \left(\begin{array}{ccc|ccc} -2 & 4 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & -1 \end{array} \right) \\
 & \begin{array}{l} r_1 - 4r_2 \rightarrow r_1 \\ r_1 - \frac{3}{2}r_3 \rightarrow r_3 \\ \frac{1}{2}r_3 + r_2 \rightarrow r_2 \end{array} \begin{array}{l} \frac{1}{2}r_3 \rightarrow r_3 \\ -\frac{1}{2}r_1 \rightarrow r_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 1 & 0 & \frac{3}{4} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right) \therefore A^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \\ \frac{3}{4} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}
 \end{aligned}$$

Underdetermined systems.

Number of unknowns is less than the number of equations.

- Are consistent but can be ~~be~~ inconsistent.
- Possible to have unique solutions.

Overdetermined system

Number of unknowns is greater than the number of equations.

- Are inconsistent.

INTRODUCTION TO MULTIVARIABLE FUNCTIONS

Quadric Surfaces

A quadric surface is the graph of any equation that can be written in the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where $A, \dots, J$ are constants.

Common Surfaces

① Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

② Elliptic Paraboloid: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

③ Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

④ Hyperboloid of two sheets: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

⑤ Elliptic Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

⑥ Hyperbolic Paraboloid: $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Functions of Several Variables

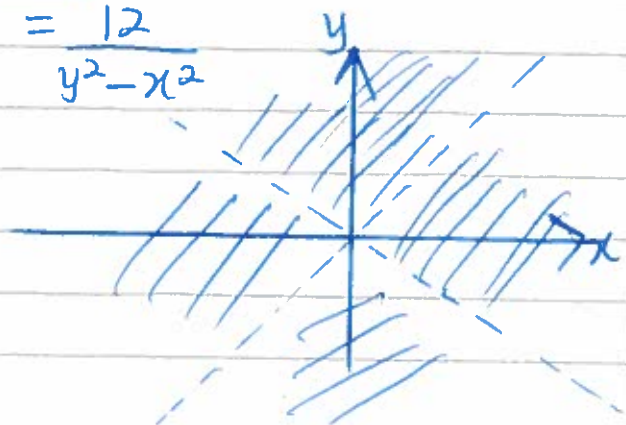
1) Find and sketch the domain of $h(x,y) = \frac{12}{y^2 - x^2}$

Soln

$$y^2 - x^2 = 0$$

$$y = \pm x$$

$$\text{Domain} = \{ (x,y) \in \mathbb{R}^2 \mid y \neq \pm x \}$$



2D-graph

$$y = f(x)$$

1 independent variable, so is its domain

3D-graph

$$z = f(x, y)$$

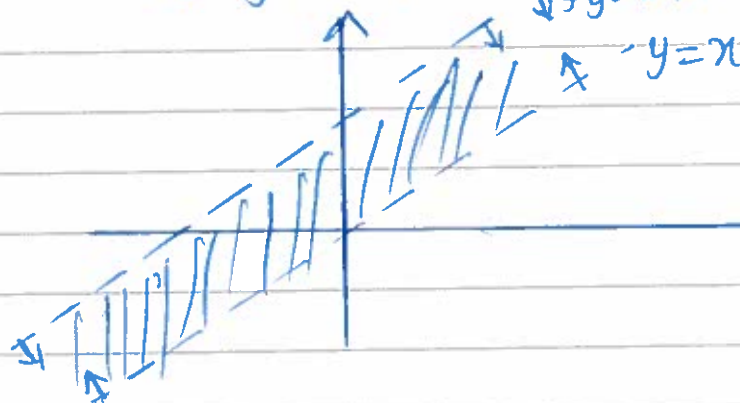
2 independent variables, so is its domain.

② Find and sketch the domain of $f(x, y) = \frac{\ln(y-x)}{\sqrt{x-y+1}}$

Solution

$$y-x > 0 \Rightarrow y > x \quad \text{--- for numerator}$$

$$x-y+1 > 0 \Rightarrow y < x+1 \quad \text{--- for denominator}$$



TRY! Find and sketch the domain of the ff

① $f(x, y) = \sqrt{x+y}$

② $f(x, y) = \sqrt{x} + \sqrt{y}$

③ $f(x, y) = \ln(9-x^2-9y^2)$

④ $f(x, y) = \sin\left(\frac{x}{y}\right)$

A Level Curve

A level curve is the shape we get when a plane intersects a surface at different levels along the axis of the dependent variable.

Derivatives of Multivariable Functions

Limits

We say, $\lim_{x \rightarrow a} f(x) = L$

$$\text{if } \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$$

Def' A function $f(x,y)$ is continuous at the point (a,b) if,
 $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$

Questions

① Determine if the ~~ff~~ limits exist or not. If they do exist give the value of the limit.

$$\textcircled{a} \lim_{(x,y) \rightarrow (5,1)} \frac{xy}{x+y}$$

$$\text{Ans!} = \frac{5 \times 1}{5+1} = \frac{5}{6} \quad \text{— Exists}$$

$$\textcircled{2} \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^4 + 3y^4}$$

Along $y=0$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^4 + 3y^4} = \lim_{(x,0) \rightarrow (0,0)} \frac{x^2 (0^2)}{x^4 + 3(0)^4} = \lim_{(x,0) \rightarrow (0,0)} 0 = 0$$

Along $x=0$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^4 + 3y^4} = \lim_{(0,y) \rightarrow (0,0)} \frac{(0)^2 y^2}{(0)^4 + 3y^4} = 0$$

Along $y=x$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^4 + 3y^4} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 \times x^2}{x^4 + 3x^4} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^4}{4x^4} = \frac{1}{4}$$

$\therefore$ the limit doesn't exist.

③ ~~lim~~ Evaluate $\lim_{(x,y) \rightarrow (-1,1)} \frac{2x^2 - xy - 3y^2}{x+y}$

Solution
 $2x^2 - xy - 3y^2 = (2x - 3y)(x + y)$

$$\begin{aligned} \lim_{(x,y) \rightarrow (-1,1)} \frac{2x^2 - xy - 3y^2}{x+y} &= \lim_{(x,y) \rightarrow (-1,1)} \frac{(2x - 3y)(x+y)}{(x+y)} \\ &= \lim_{(x,y) \rightarrow (-1,1)} 2x - 3y \\ &= 2(-1) - 3(1) \\ &= \underline{\underline{-5}} \end{aligned}$$

Evaluate: $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{x - \sqrt{xz} - \sqrt{xy} + \sqrt{yz}}{x - \sqrt{xz} + \sqrt{xy} - \sqrt{yz}}$

Soln
 $x - \sqrt{xz} - \sqrt{xy} + \sqrt{yz} = \sqrt{x}(\sqrt{x} - \sqrt{z}) - \sqrt{y}(\sqrt{x} - \sqrt{z})$
 $= (\sqrt{x} - \sqrt{y})(\sqrt{x} - \sqrt{z})$
 $x - \sqrt{xz} + \sqrt{xy} - \sqrt{yz} = \sqrt{x}(\sqrt{x} - \sqrt{z}) + \sqrt{y}(\sqrt{x} - \sqrt{z})$
 $= (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{z})$

$$\begin{aligned} &= \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{(\sqrt{x} - \sqrt{y})(\sqrt{x} - \sqrt{z})}{(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{z})} = \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} \\ &= \frac{\sqrt{1} - \sqrt{1}}{\sqrt{1} + \sqrt{1}} \\ &= \frac{0}{2} \\ &= \underline{\underline{0}} \end{aligned}$$

Say we have $f(x,y)$

$$f_x(x,y) = \lim_{h \rightarrow 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \rightarrow 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Example

Find f_x and f_y of $f(x,y) = x^2 + 2xy + y^3$

Soln

$$f_x(x,y) = \lim_{h \rightarrow 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2 + 2(x+h)y + y^3 - (x^2 + 2xy + y^3)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 2hy}{h}$$

$$= \lim_{h \rightarrow 0} 2x + h + 2y$$

$$\therefore f_x(x,y) = \underline{\underline{2x + 2y}}$$

$$f_y(x,y) = \lim_{h \rightarrow 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + 2x(y+h) + (y+h)^3 - (x^2 + 2xy + y^3)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2xh + 3hy^2 + 3h^2y + h^3}{h}$$

$$= \lim_{h \rightarrow 0} 2x + 3y^2 + 3hy + h^2$$

$$\therefore f_y(x,y) = \underline{\underline{2x + 3y^2}}$$

Qn: Find all the first order partial derivatives for the ff

① $f(x,y) = e^{x^2+3y}$
 $f_x(x,y) = \frac{\partial F}{\partial x} = 2x e^{x^2+3y}$

$$f_y(x,y) = \frac{\partial F}{\partial y} = 3e^{x^2+3y}$$

② $F(x,y) = \ln(1+e^{-xy})$ at $(2,-4)$

$$f_x = \frac{\partial F}{\partial x} = \frac{\frac{\partial \ln}{\partial x} \cdot e^{-xy}}{1+e^{-xy}} = \frac{-y \cdot e^{-xy}}{1+e^{-xy}}$$

$$\left. \frac{\partial F}{\partial x} \right|_{\substack{x=2 \\ y=-4}} = \underline{\underline{3.9986}}$$

$$f_y = \frac{\partial F}{\partial y} = \frac{-x e^{-xy}}{1+e^{-xy}}$$

$$\left. \frac{\partial F}{\partial y} \right|_{\substack{x=2 \\ y=-4}} = \underline{\underline{-1.9993}}$$

③ $f(x,y) = \frac{x^2}{x^2+y^2}$

$$f_x = \frac{2xy^2}{(x^2+y^2)^2}$$

$$f_y = \frac{-2x^2y}{(x^2+y^2)^2}$$

IRY: Find $\frac{\partial Z}{\partial x}$ and $\frac{\partial Z}{\partial y}$ for the ff

① $x^3z^2 - 5xy^5z = x^2+y^3$

② $x^2 \sin(2y-5z) = 1 + y \cos(6zx)$

Question

Determine if $f(x,y) = \frac{x^2}{y^3}$ is increasing or decreasing at $(2,5)$

(a) if we allow x to vary and hold y fixed.

(b) if we allow y to vary and hold x fixed.

Soln

$$(a) f_x = \frac{2x}{y^3}$$

$$f_x(2,5) = \frac{4}{125} > 0 \text{ — Increasing}$$

$$(b) f_y(x,y) = -\frac{3x^2}{y^4}$$

$$f_y(2,5) = -\frac{12}{625} < 0 \text{ — Decreasing}$$

Question

(1) $P(u,v) = \ln(u^2 + v^2 + 4)$ find $\frac{\partial^2 P}{\partial u \partial v}$ and $\frac{\partial^2 P}{\partial v \partial u}$

$$\frac{\partial P}{\partial v} = \frac{2v}{u^2 + v^2 + 4} \quad \Rightarrow \quad \frac{\partial P}{\partial u \partial v} = \frac{-4uv}{(u^2 + v^2 + 4)^2}$$

$$\frac{\partial P}{\partial u} = \frac{2u}{u^2 + v^2 + 4} \quad \Rightarrow \quad \frac{\partial P}{\partial v \partial u} = \frac{-4uv}{(u^2 + v^2 + 4)^2}$$

> Clairaut's Theorem

Clairaut's Theorem

Suppose that f is defined on a disk D that contains (a,b) . If the functions f_{xy} and f_{yx} are continuous on this disk then,

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Qn: Verify Clairaut's theorem for $f(x,y) = xe^{-x^2y^2}$

$$f_n(x,y) = e^{-x^2y^2}$$

$$f_x(x,y) = e^{-x^2y^2} - 2x^2y^2e^{-x^2y^2}$$

$$f_y(x,y) = -2x^2ye^{-x^2y^2}$$

$$f_{xy} = -6x^2ye^{-x^2y^2} + 4x^4y^3e^{-x^2y^2}$$

$$f_{yx} = -6yx^2e^{-x^2y^2} + 4y^3x^4e^{-x^2y^2}$$

CHAIN RULE

$$z = f(x,y)$$

z - dependent variable

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

(x,y) - intermediate //

t - independent //

Qn:

If $z = x^2y - xy^3$, $x = \cos t$, $y = e^t$
find $\frac{dz}{dt}$

Soln

$$\frac{\partial z}{\partial x} = 2xy - y^3$$

$$\frac{dx}{dt} = -\sin t$$

$$\frac{\partial z}{\partial y} = x^2 - 3xy^2$$

$$\frac{dy}{dt} = e^t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$= (2xy - y^3)(-\sin t) + (x^2 - 3xy^2)(e^t)$$

$$\left. \frac{dz}{dt} \right|_{t=0} = x^2 - 3xy^2$$

$$= (1)^2 - 3(1)(1)^2$$

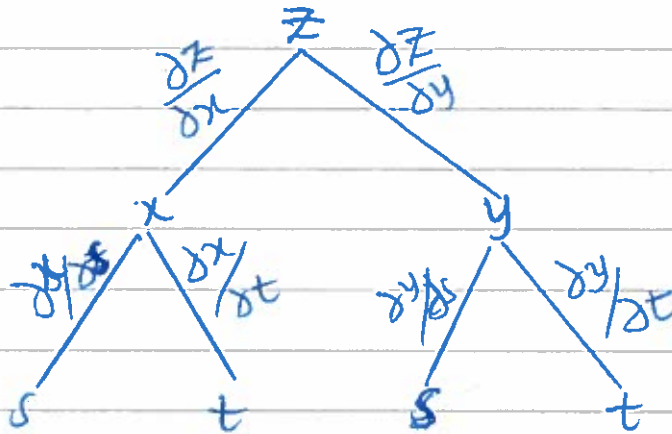
$$= -2$$

≡

Tree Diagram

Given that $z = f(x, y)$, $x = g(s, t)$, $y = h(s, t)$
find: $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

Soln



~~$\frac{\partial z}{\partial t}$~~

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial t}$$

Directional Derivatives

Finding the rate of change of $f(x,y)$ if we allow both x and y to change simultaneously.

The rate of change of $f(x,y)$ in the direction of the unit vector $\vec{u} = \langle u_1, u_2 \rangle$ is called the directional derivative and is denoted by $\text{Dir} f(x,y)$. The definition is,

$$\text{Dir} f(x,y) = \lim_{t \rightarrow 0} \frac{f(x+u_1 t, y+u_2 t) - f(x,y)}{t}$$

How to Compute

$\text{Dir} f(x,y)$ is a directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$. $f(x_0+u_1 t, y_0+u_2 t)$ is the value of f along L

$$g(t) = f(x_0+u_1 t, y_0+u_2 t)$$

$$g'(t) = \text{Dir} f(x_0, y_0)$$

$$g'(t) = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt}$$

$$g'(t) = \text{Dir} f(x,y) = \frac{\partial F}{\partial x} u_1 + \frac{\partial F}{\partial y} u_2$$

Question

① Find the derivative of $f(x,y) = x^3 - 2x^2 + y^3$ at $P(1,2)$ in the direction of a vector that makes an angle of $\theta = \pi/6$ with the x -axis.

Solution

$$\vec{v} = \langle \cos(\pi/6), \sin(\pi/6) \rangle$$

$$\vec{v} = \langle \sqrt{2}/2, 1/2 \rangle$$

$$\frac{\partial F}{\partial x} = 3x^2 - 4x$$

$$\frac{\partial F}{\partial y} = 3y^2$$

$$\text{Dir} f(x,y) \Big|_{\substack{x=1 \\ y=2}} = (3x^2 - 4x) \left(\frac{\sqrt{2}}{2} \right) + 3y^2 \left(\frac{1}{2} \right)$$

$$= 6 - \frac{\sqrt{3}}{2}$$

for $f(x, y, z)$ and $\vec{u} = \langle u_1, u_2, u_3 \rangle$ - unit vector.

$$\begin{aligned} D_{\vec{u}} f(x, y, z) &= f_x(x, y, z)u_1 + f_y(x, y, z)u_2 + f_z(x, y, z)u_3 \\ &= \langle f_x, f_y, f_z \rangle \cdot \langle u_1, u_2, u_3 \rangle \end{aligned}$$

Question

Find $D_{\vec{u}} f(x, y, z)$ where $f(x, y, z) = x^2z + y^3z^2 - xyz$ in the direction of $\vec{v} = \langle -1, 0, 3 \rangle$

solution

$$\text{unit vector, } \vec{v} = \frac{1}{|\vec{v}|} \langle -1, 0, 3 \rangle = \frac{1}{\sqrt{10}} \langle -1, 0, 3 \rangle = \left\langle -\frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}} \right\rangle$$

$$f_x = 2xz - yz \quad f_y = 3y^2z^2 - xz \quad f_z = (x^2 + 2y^3z - xy)$$

$$\begin{aligned} D_{\vec{u}} f(x, y, z) &= (2xz - yz) \left(-\frac{1}{\sqrt{10}} \right) + (3y^2z^2 - xz)(0) + (x^2 + 2y^3z - xy) \left(\frac{3}{\sqrt{10}} \right) \\ &= \frac{1}{\sqrt{10}} (3x^2 + 6y^3z - 3xy - 2xz + yz) \end{aligned}$$

Gradient Vectors

$$D_{\vec{u}} f(x, y) = \frac{\partial F}{\partial x} u_1 + \frac{\partial F}{\partial y} u_2$$

$$= \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle \cdot \langle u_1, u_2 \rangle$$

$$= \vec{\nabla} f \cdot \vec{u} \quad \text{--- unit vector } \vec{u}$$

$\vec{\nabla} f$ is called gradient vector. $\text{grad}(f)$

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = |\vec{\nabla} f| |\vec{u}| \cos \theta$$

$$= |\vec{\nabla} f| \cos \theta \quad |\vec{u}| = 1 \quad \text{--- unit vector}$$

① When $\theta = 0 \Rightarrow \cos \theta = 1$

$$D_{\vec{u}} f = |\vec{\nabla} f| \quad \text{--- maximum slope}$$

② When $\theta = 180^\circ \cos \theta = -1$

$$D_{\vec{u}} f = -|\vec{\nabla} f|$$

