

POLAR COORDINATE SYSTEM

1

Conversion from Cartesian to polar

$$(x, y) \rightarrow (r, \theta)$$

$$r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

From Polar to Cartesian

$$(r, \theta) \rightarrow (x, y)$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Example

Convert $(-4, 2\pi/3)$ into Cartesian coordinates

Solution

$$x = -4 \cos(2\pi/3) = 2$$

$$y = -4 \sin(2\pi/3) = -2\sqrt{3}$$

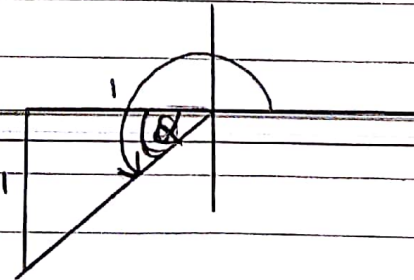
$$\therefore (-4, 2\pi/3) \rightarrow (2, -2\sqrt{3})$$

2) Convert $(-1, -1)$ into polar coordinates

Solution

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

To solve for θ ,



$$\alpha = \tan^{-1}(1/1)$$

$$\theta = \pi + \alpha$$

$$\alpha = \pi/4$$

$$\theta = \pi + \pi/4 \\ = 5\pi/4$$

$$\therefore (-1, -1) = (\sqrt{2}, \frac{5\pi}{4})$$

③ Convert the polar equation $r = 6 \sin \theta$ to cartesian and describe the figure.

Solution

$$r = 6 \sin \theta$$

$$r^2 = 6 r \sin \theta$$

$$\text{but } r^2 = x^2 + y^2 \quad y = r \sin \theta$$

$$\Rightarrow x^2 + y^2 = 6y$$

$$x^2 + y^2 - 6y = 0$$

By completing the squares

$$(x-0)^2 + (y-3)^2 = 3^2$$

This is a circle with centre $(0, 3)$ and radius, $r = 3$.

TRY

Convert the ff polar equations into cartesian and describe the cartesian.

$$\textcircled{1} \quad r = 2 \sin \theta + 2 \cos \theta$$

$$\textcircled{2} \quad r = -8 \cos \theta$$

Cylindrical Coordinates

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It is the polar coordinates with the z -axis added (r, θ, z) where $r \geq 0$.

Cartesian to Cylindrical

$$(x, y, z) \rightarrow (r, \theta, z)$$

$$r^2 = x^2 + y^2 \quad \theta = \tan^{-1}(y/x) \quad z = z$$

Cylindrical to Cartesian

$$(r, \theta, z) \rightarrow (x, y, z)$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

TRY

(a) Convert the ff from (r, θ, z) to (x, y, z)

(i) $(3, -\pi/6, 2)$

(ii) $(2, 2\pi/3, 1)$

(b) Convert the ff from (x, y, z) to (r, θ, z)

(i) $(\sqrt{2}, -\sqrt{2}, 4)$

(ii) $(2, -3, -7)$

(c) Identify each surface

(a) $r = 5$

(b) $x^2 + z^2 = 100$

(c) $z = r$

Spherical Coordinates

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Rectangular to Spherical

$$(x, y, z) \rightarrow (p, \phi, \theta)$$

$$p^2 = x^2 + y^2 + z^2$$

$$\tan \theta = \frac{y}{x}$$

$$\cos \phi = \frac{z}{p}$$

Spherical to Rectangular

$$(p, \phi, \theta) \rightarrow (x, y, z)$$

$$x = p \sin \phi \cos \theta$$

$$y = p \sin \phi \sin \theta$$

$$z = p \cos \phi$$

⌘

NOTE

① $p \geq 0$

② $0 \leq \phi \leq \pi$

③ If $\phi \leq \pi/2$, the z-value is positive

④ If $\phi > \pi/2$, " " " negative.

Spherical to Cylindrical

$$(p, \phi, \theta) \rightarrow (r, \theta, z)$$

$$z = p \cos \phi$$

$$r = p \sin \phi$$

$$\theta = \theta$$

Cylindrical to Spherical

$$(r, \theta, z) \rightarrow (p, \phi, \theta)$$

$$p^2 = r^2 + z^2$$

$$\theta = \theta$$

$$\cos \phi = \frac{z}{p}$$

Example.

① $(x, y, z) \rightarrow (p, \phi, \theta)$

(a) $(\frac{3}{2}, \frac{3}{2}, -\frac{3}{\sqrt{2}})$

(b) $(0, 2\sqrt{3}, -2)$

② $(p, \phi, \theta) \rightarrow (x, y, z)$

(a) $(2, \pi/4, \pi/3)$

(b) $(3, \frac{\pi}{4}, \frac{3\pi}{4})$

MULTIPLE INTEGRALS

5

Double Integral

The double integral of 'f' over the rectangle R is

$$\iint_R f(x,y) dA = \lim_{\Delta \rightarrow 0} \sum_{k=1}^n f(x_k^*, y_k^*) \Delta A_k$$

$$\text{where } \Delta A_k = \Delta x_k^* \Delta y_k^*$$

If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y) dA$$

Iterated Integrals

Fubini's Theorem;

If $f(x,y)$ is continuous on the rectangle

$R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$ or $R = [a,b] \times [c,d]$ then,

$$\iint_R f(x,y) dA = \int_a^b \int_c^d f(x,y) dy dx = \int_c^d \int_a^b f(x,y) dx dy$$

These integrals are called iterated integrals.

Examples

Solve $\iint_R x^2 y dA$ over $R = [0,3] \times [1,2]$.

Solution

$$\iint_R x^2 y dA = \int_1^2 \int_0^3 x^2 y dx dy$$

$$= \int_1^2 \left[\frac{x^3 y}{3} \right]_0^3 dy$$

$$= \int_1^2 9y dy$$

$$= \frac{27}{2}$$

Find the volume of the solid S that is bounded by the elliptic paraboloid ~~and~~ $x^2 + y^2 + z = 16$, the planes $x=2$ and $y=2$ and the Three Coordinates planes.

Solution

$$z = f(x, y) = 16 - x^2 - y^2$$

$$R = [0, 2] \times [0, 2]$$

$$\begin{aligned} V &= \int_0^2 \int_0^2 (16 - x^2 - y^2) dx dy \\ &= \int_0^2 \left(16x - \frac{x^3}{3} - xy^2 \right) \Big|_0^2 dy \\ &= \int_0^2 \left(\frac{88}{3} - 2y^2 \right) dy \\ &= \left(\frac{88y}{3} - \frac{2}{3}y^3 \right) \Big|_0^2 \\ &= \frac{160}{3} \text{ cubic units.} \end{aligned}$$

TRY

1) Solve $\iint_R (x - 3y^2) dA$ over $R = [0, 2] \times [1, 2]$

2) $\iint_R y \sin(xy) dA$, where $R = [1, 2] \times [0, \pi]$

3) $\iint_R y e^{xy} dA$, $R = [0, 1] \times [0, 2]$

4) $\iint_R \sin x \cos y dA$ $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$

5) $\iint_R \frac{1}{(2x+3y)^2} dA$, $R = [0, 1] \times [1, 2]$

Fact

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If $f(x,y) = g(x)h(y)$ and we are iterating over $R = [a,b] \times [c,d]$ then,

$$\begin{aligned}\iint_R f(x,y) dA &= \iint_R g(x)h(y) dA \\ &= \left(\int_a^b g(x) dx \right) \left(\int_c^d h(y) dy \right)\end{aligned}$$

Example

Evaluate $\iint_R x \cos^2 y dA$ $R = [-2,3] \times [0, \pi/2]$

Soln

$$\begin{aligned}\iint_R x \cos^2 y dA &= \left(\int_{-2}^3 x dx \right) \left(\int_0^{\pi/2} \cos^2 y dy \right) \\ &= \left(\frac{1}{2} x^2 \Big|_{-2}^3 \right) \left(\int_0^{\pi/2} \frac{1}{2} (1 + \cos(2y)) dy \right) \\ &= \frac{5\pi}{8}\end{aligned}$$

Average Value

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$$f_{\text{av}} = \frac{1}{\text{Area of } R} \iint_R f(x, y) dA$$

Example:

Find the average value of the quantity $2-x-y$ over the square $R = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 2\}$

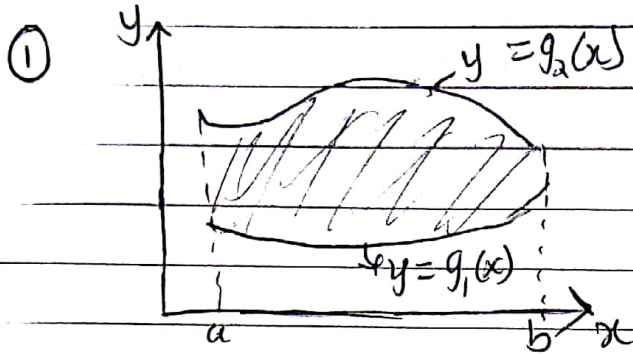
Soln

$$\text{Area of } R = 2 \times 2 = 4$$

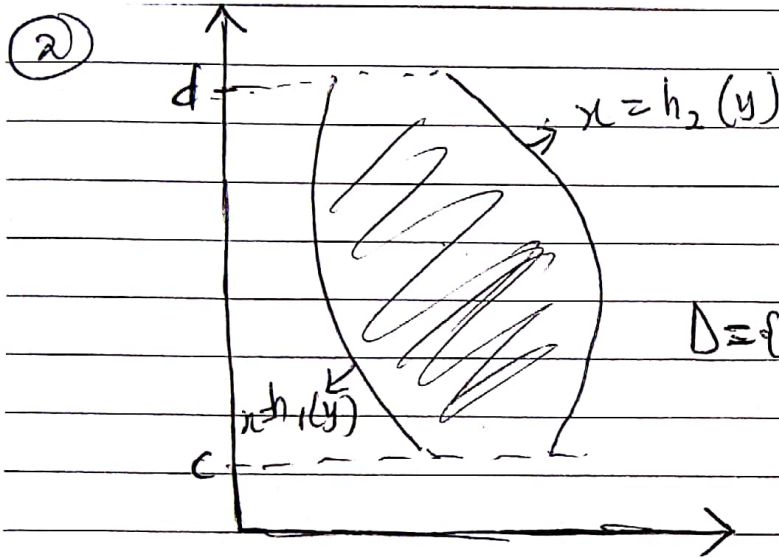
$$\begin{aligned} f_{\text{av}} &= \frac{1}{4} \int_0^2 \int_0^2 (2-x-y) dx dy \\ &= \frac{1}{4} \int_0^2 \left(2x - \frac{x^2}{2} - xy \right) \Big|_0^2 dy \\ &= \frac{1}{4} \int_0^2 (2 - 2y) dy \\ &= \frac{1}{4} (2y - y^2) \Big|_0^2 \\ &= \underline{\underline{0}} \end{aligned}$$

Double Integrals Over General Regions

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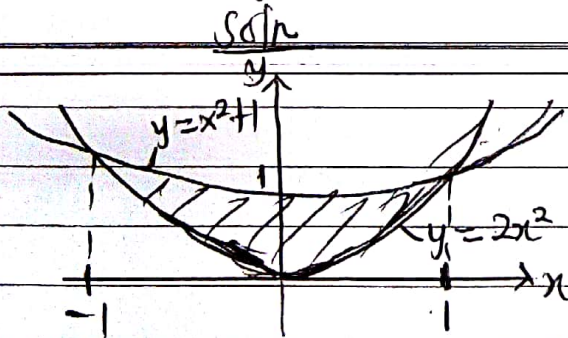
The region is
 $D = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$



The region is
 $D = \{(x, y) : h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$

Example

Evaluate $\iint_D (x+2y) dA$ where D is the region bounded by the parabola $y = 2x^2$ and $y = 1+x^2$



Point of intersection

$$2x^2 = x^2 + 1$$

$$x = \pm 1$$

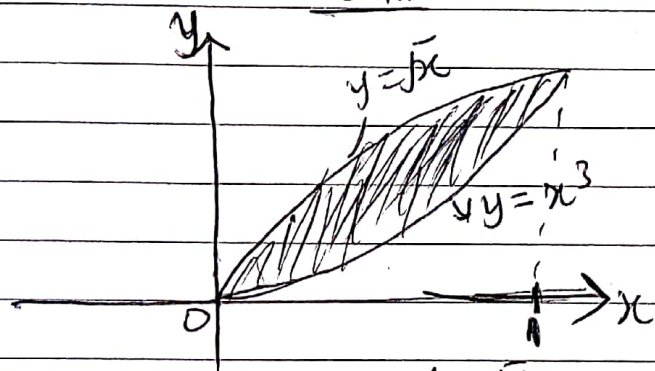
Region, $D = \{(x, y) : -1 \leq x \leq 1, 2x^2 \leq y \leq x^2 + 1\}$

$$\Rightarrow \iint_D (x+2y) dA = \int_{-1}^1 \int_{2x^2}^{x^2+1} (x+2y) dy dx$$

$$\begin{aligned}
 &= \int_{-1}^1 xy + y^2 \Big|_{2x}^{x^2+1} dx \\
 &= \int_{-1}^1 -3x^4 - x^3 + 2x^2 + x + 1 dx \\
 &= \frac{32}{15}
 \end{aligned}$$

② Evaluate $\iint_D 4xy - y^3 dA$, D is the region bounded by $y = \sqrt{x}$ and $y = x^3$.

Soln

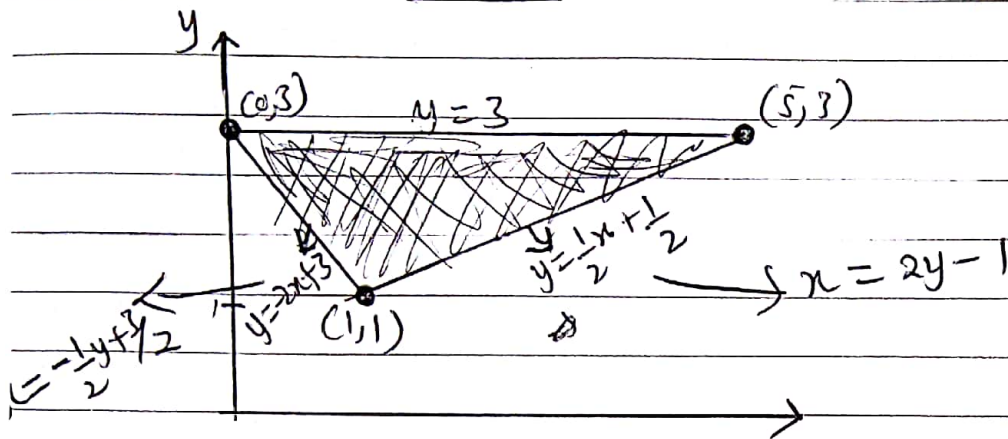


$$\begin{aligned}
 x^3 - \sqrt{x} &= 0 \\
 \sqrt{x}(x^{3/2} - 1) &= 0 \\
 x &= 0, x = 1
 \end{aligned}$$

$$\begin{aligned}
 \iint_D 4xy - y^3 dA &= \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^3 dy dx \\
 &= \int_0^1 \frac{7}{4}x^2 - 2x^7 + \frac{1}{4}x^{12} dx \\
 &= \frac{55}{156}
 \end{aligned}$$

3) $\iint_D 6x^2 - 40y \, dA$, where D is the triangle with vertices $(0,3)$, $(1,1)$ and $(5,3)$.

Solution



$$D = \{(x,y) : -\frac{1}{2}y + \frac{3}{2} \leq x \leq 2y - 1, 1 \leq y \leq 3\}$$

$$\begin{aligned} \iint_D 6x^2 - 40y \, dA &= \int_1^3 \int_{-\frac{1}{2}y + \frac{3}{2}}^{2y-1} 6x^2 - 40y \, dx \, dy \\ &= -\frac{935}{2} \end{aligned}$$

TRY

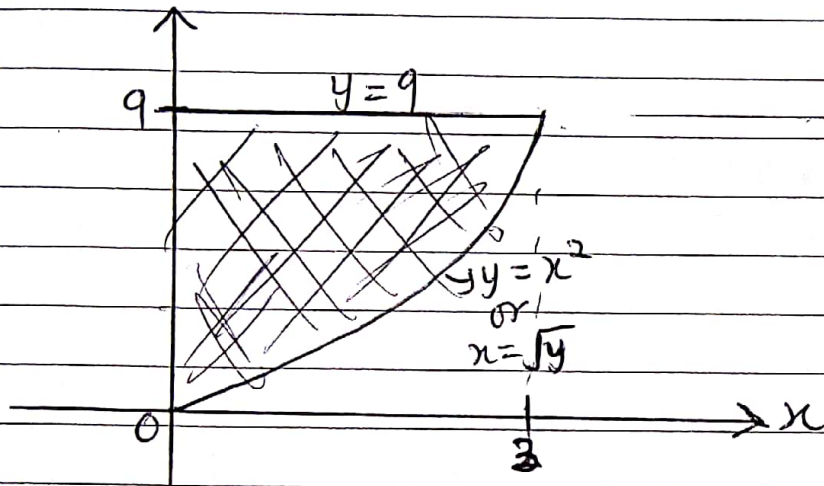
Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy -plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

2) Evaluate $\iint_D e^{xy} \, dA$, $D = \{(x,y) : 1 \leq y \leq 2, y \leq x \leq y^3\}$

① Evaluate $\int_0^3 \int_{x^2}^9 x^3 e^{y^3} dy dx$ by first reversing the order of integration.

Solution

The region is $0 \leq x \leq 3, x^2 \leq y \leq 9$



New region interval

$$0 \leq y \leq 9$$

$$0 \leq x \leq \sqrt{y}$$

$$\Rightarrow \int_0^3 \int_{x^2}^9 x^3 e^{y^3} dy dx = \int_0^9 \int_0^{\sqrt{y}} x^3 e^{y^3} dx dy$$

$$= \frac{1}{12} (e^{729} - 1)$$

TRY

valuate $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4 + 1} dx dy$ by first reversing

Double Integrals in Polar Coordinates

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$$dA = r dr d\theta$$

$$\iint_D f(x,y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$

$$x = r \cos \theta \quad r^2 = x^2 + y^2$$

$$y = r \sin \theta$$

Example

Evaluate $\iint_D e^{x^2+y^2} dA$ by converting to polar coordinates where D is the unit circle centered at the origin.

Soln

Unit circle

$$0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$\iint_D e^{x^2+y^2} dA = \int_0^{2\pi} \int_0^1 e^{r^2} \cdot r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 r e^{r^2} dr d\theta$$

$$= \pi(e-1)$$

TRY

Evaluate $\iint_D xy dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

- (2) Determine the area of the region that lies inside $r = 3 + 2 \sin \theta$ and outside $r = 2$.

Triple Integrals

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$$\iiint_D f(x,y,z) dV = \int_a^b \int_{g(x)}^{h(x)} \int_{q(x)}^{r(x)} f(x,y,z) dz dy dx.$$

Example

Find the volume of the prism D in the first octant bounded by the planes $y = 4 - 2x$ and $z = 6$

Solution

$$x=0, y=0, z=0 \quad \text{when } y=0, \Rightarrow x=2$$
$$x=2, y=4-2x, z=6$$

$$\begin{aligned} \text{Volume, } V &= \int_0^6 \int_0^2 \int_0^{4-2x} dz dx dy \\ &= \int_0^6 \int_0^2 (4-2x) dx dz \\ &= \int_0^6 \left(4x - x^2 \Big|_0^2 \right) dz \\ &= \int_0^6 4 dz \\ &= 24 \text{ cubic units.} \end{aligned}$$

TRY

1) Evaluate the integral $\iiint_D xyz^2 dV$ where D is the rectangular box given by $D = \{(x,y,z): 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$

2) A solid box D is bounded by the planes $x=0, x=3, y=0, y=2, z=0$ and $z=1$. The density of the box decreases linearly in the positive z -direction and is given by $f(x,y,z) = 2-z$. Find the mass of the box.

NB: The volume of a three-dimensional region E is given by the integral $V = \iiint_E dV$

Triple Integrals in Cylindrical Coordinates

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$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

$$dV = r \, dz \, dr \, d\theta$$

$$\iiint_E f(x, y, z) \, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r, \theta)}^{u_2(r, \theta)} r f(r \cos \theta, r \sin \theta, z) \, dz \, dr \, d\theta$$

Example

evaluate $\iiint_E y \, dV$ where E is the region that lies below the plane $z = x + 2$ above the xy -plane and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

Soln

~~$z=0$~~ Conversion for the z -coordinate

$$0 \leq z \leq x + 2 \Rightarrow 0 \leq z \leq r \cos \theta + 2$$

Is the region b/n the two circles

$$\Rightarrow 0 \leq \theta \leq 2\pi \quad \text{and} \quad 1 \leq r \leq 2$$

$$\Rightarrow \iiint_E y \, dV = \int_0^{2\pi} \int_1^2 \int_0^{r \cos \theta + 2} (r \sin \theta) r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_1^2 \frac{1}{2} r^3 \sin(2\theta) + 2r^2 \sin \theta \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{8} r^4 \sin(2\theta) + \frac{2}{3} r^3 \sin \theta \right) \Big|_1^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{15}{8} \sin(2\theta) + \frac{14}{3} \sin \theta \, d\theta$$

$$= \underline{\underline{0}}$$

TRY

Convert $\int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy$ into an ~~the~~ integral

in cylindrical coordinates.

Triple Integrals in Spherical Coordinates 16

$$(x, y, z) \rightarrow (\rho, \phi, \theta)$$

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

$$\rho^2 = x^2 + y^2 + z^2$$

$$dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

$$\iiint_E f(x, y, z) \, dV = \int_a^b \int_\alpha^\beta \int_\delta^\gamma \rho^2 \sin \phi \, f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \, d\rho \, d\theta \, d\phi$$

Example

Evaluate $\iiint_E 16z \, dV$ where E is the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

Solution

$$0 \leq \rho \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \pi/2$$

$$\Rightarrow \iiint_E 16z \, dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 \rho^2 \sin \phi (16\rho \cos \phi) \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 8\rho^3 \sin(2\phi) \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi/2} \int_0^{2\pi} 2 \sin(2\phi) \, d\theta \, d\phi$$

$$= \underline{\underline{4\pi}}$$

TRY

Evaluate $\iiint z \, dV$ where E lies between $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first quadrant.

Hint: $1 \leq \rho \leq 2$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \pi/2$$

Transformation in the Plane

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It relates pairs of variables $(x, y) = T(u, v)$ where $T: x = g(u, v)$ and $y = h(u, v)$

Example

Determine the new region that we get by applying the given transformation to the region.

R is the ellipse $x^2 + \frac{y^2}{36} = 1$ and the transformation is

$$x = \frac{u}{2} \text{ and } y = 3v$$

$$x^2 + \frac{y^2}{36} = 1$$

$$\left(\frac{u}{2}\right)^2 + \frac{(3v)^2}{36} = 1$$

$$\Rightarrow u^2 + v^2 = 1$$

Jacobian Transformation

The Jacobian of the transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \cdot \frac{\partial x}{\partial v}$$

Suppose we want to integrate $f(x, y)$ over the region R . Under the transformation $x = g(u, v)$, $y = h(u, v)$ the region becomes S and the integral becomes.

$$\iint_R f(x, y) dA = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

$$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

Example

Compute the Jacobian of the transformation

$$T: x = g(r, \theta) = r \cos \theta$$

$$y = h(r, \theta) = r \sin \theta$$

Soln

$$\frac{\partial x}{\partial r} = \cos \theta \quad \frac{\partial x}{\partial \theta} = -r \sin \theta$$

$$\frac{\partial y}{\partial r} = \sin \theta \quad \frac{\partial y}{\partial \theta} = r \cos \theta$$

$$\begin{aligned} J(r, \theta) &= \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\ &= r \cos^2 \theta + r \sin^2 \theta \\ &= r \end{aligned}$$

(2) Evaluate the integral $\iint_R \frac{x-2y}{3x-y} dA$ where R is the parallelogram enclosed by the lines $x-2y=0$, $x-2y=4$, $3x-y=1$ and $3x-y=8$

Soln

$$\text{let } u = x-2y \text{ and } v = 3x-y$$

$$x-2y=0 \Rightarrow u=0$$

$$3x-y=1 \Rightarrow v=1$$

$$x-2y=4 \Rightarrow u=4$$

$$3x-y=8 \Rightarrow v=8$$

$$u = x-2y$$

$$v = 3x-y$$

$$x = u+2y$$

$$y = 3x-v$$

$$\Rightarrow x = u+2(3x-v)$$

$$y = 3\left(\frac{2v-u}{5}\right) - v$$

$$\Rightarrow x = \frac{2v-u}{5}$$

$$y = \frac{1v-3u}{5}$$

$$\frac{\partial x}{\partial u} = -\frac{1}{5} \quad \frac{\partial x}{\partial v} = \frac{2}{5}$$

$$\frac{\partial y}{\partial u} = -\frac{3}{5} \quad \frac{\partial y}{\partial v} = \frac{1}{5}$$

$$J(u,v) = \begin{vmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{vmatrix} = \frac{1}{5}$$

$$\iint_R \frac{x-2y}{3x-y} dA = \int_0^4 \int_1^8 \frac{u}{v} \left| \frac{1}{5} \right| dv du = 3.327$$

Example 2

$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$ where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, $(0,1)$.

Soln

let $u = y - x$ — (1) and $v = y + x$ — (2)

$(x,y) \rightarrow (u,v)$ (1) + (2) and (2) - (1)

$(1,0) \rightarrow (-1,1)$ $x = \frac{1}{2}(v-u)$ and $y = \frac{1}{2}(u+v)$

$(2,0) \rightarrow (-2,2)$

$(0,2) \rightarrow (2,2)$ $-2 \leq u \leq 2$ and $1 \leq v \leq 2$

$(0,1) \rightarrow (1,1)$

$$\frac{\partial x}{\partial u} = -\frac{1}{2} \quad \frac{\partial x}{\partial v} = \frac{1}{2} \quad \frac{\partial y}{\partial u} = \frac{1}{2} \quad \frac{\partial y}{\partial v} = \frac{1}{2}$$

$$J(u,v) = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\frac{1}{2}$$

$$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA = \int_{-2}^2 \int_1^2 \cos\left(\frac{u}{v}\right) \left| -\frac{1}{2} \right| dv du$$

Example

Let R be the region in the first quadrant bounded by the parabolas $x = y^2$, $x = y^2 - 4$, $x = 9 - y^2$ and $x = 16 - y^2$. Evaluate $\iint_R y^2 dA$

Soln

$$x - y^2 = 0 \quad \Rightarrow u = 0$$

$$u = x - y^2 \quad \text{--- (1)}$$

$$x - y^2 = -4 \quad u = -4$$

$$v = x + y^2 \quad \text{--- (2)}$$

$$x + y^2 = 9 \quad v = 9$$

$$\text{(1)+(2) and (4)-(2)}$$

$$x + y^2 = 16 \quad v = 16$$

$$x = \frac{1}{2}(u+v)$$

$$y^2 = \frac{1}{2}(v-u)$$

$$\frac{\partial x}{\partial u} = \frac{1}{2} \quad \frac{\partial x}{\partial v} = \frac{1}{2}$$

$$\frac{\partial y}{\partial u} = -\frac{1}{2\sqrt{v-u}} \quad \frac{\partial y}{\partial v} = \frac{1}{2\sqrt{v-u}}$$

$$J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2\sqrt{v-u}} & \frac{1}{2\sqrt{v-u}} \end{vmatrix}$$

$$= \frac{1}{2\sqrt{v-u}} \cdot \frac{1}{\sqrt{v-u}}$$

$$\iint_R y^2 dA = \int_{-4}^9 \int_0^{16} \frac{1}{2} (v-u) \cdot \left| \frac{1}{2\sqrt{v-u}} \cdot \frac{1}{\sqrt{v-u}} \right| dv du$$

$$= 106.27099$$

Vector Fields

21

A vector field is a function $\vec{F}$ that assigns to each point (x, y) or (x, y, z) a vector $\vec{F}(x, y)$ or $\vec{F}(x, y, z)$.

If $\vec{F}$ is a conservative vector field, there is a scalar function ϕ such that $\vec{F} = \nabla\phi$ and ϕ is called potential function.

Generally, $\vec{F} = \langle f, g, h \rangle$ in 3D. where

$$f = f(x, y, z) \quad g = g(x, y, z) \quad \text{and} \quad h = h(x, y, z)$$

Suppose $\vec{F}$ is conservative, then

$f_y = g_x$
$f_z = h_x$
$g_z = h_y$

Example

Determine if the ff are conservative.

① $\vec{F} = \langle 2x^3 + xy^2, 2y^3 + x^2y \rangle$

$$f_y = 2xy \quad g_x = 2xy$$

$\therefore \vec{F}$ is conservative.

② $\vec{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2yz \rangle$

$$f_y = 2x \quad g_x = 2x$$

$$f_z = -2z \quad h_x = 0$$

$$g_z = 2 \quad h_y = 2 - 2z$$

$\therefore \vec{F}$ is not conservative.

How to Find the Potential Function (ϕ)

① Find $\phi = \int \phi_x dx + C(y, z)$

② Compute ϕ_y and equate result to g to obtain an expression for $C_y(y, z)$

③ Find $\int C_y(x, z) dy$ to obtain $C(y, z)$ including an arbitrary $d(z)$.

④ Compute ϕ_z and equate result to get $d(z)$

A similar procedure works for ϕ_y and ϕ_z in step ①.

Example

22

Find the potential function if $\vec{F} = \nabla \phi$ and
 $\vec{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle$

Soln

~~$\phi = \int$~~

Check whether $\vec{F}$ is conservative. If $\vec{F}$ is conservative, then compute the potential function, ϕ

$$\phi = \int 2xy - z^2 dx$$
$$= x^2 y - xz^2 + (y, z)$$

$$\phi_y = x^2 + (y, z)$$

$$x^2 + (y, z) = x^2 + 2z$$

$$(y, z) = 2z$$

$$C(y, z) = \int (y, z) dy = \int 2z dy$$
$$= 2yz + d(z)$$

$$\therefore \phi = x^2 y - xz^2 + 2yz + d(z)$$

$$\phi_z = -2xz + 2y + d'(z)$$

$$-2xz + 2y + d'(z) = 2y - 2xz$$

$$d'(z) = 0$$

$$d(z) = \int d'(z) dz$$

$$= \int 0 dz = C \text{ (a constant)}$$

$$\therefore \phi = x^2 y - xz^2 + 2yz + C$$

TRY

Find the potential function of

$$\textcircled{1} \vec{F} = \langle x^3 - xy, -\frac{x^2}{2} + y \rangle$$

$$\textcircled{2} \vec{F} = \langle x^3, 2y, z^3 \rangle$$

Divergence and Curl

23

Divergence is a measure of contraction and expansion of a vector field around the neighbourhood of a point P in the field.

Computing Divergence

Given $\vec{F}$, the divergence of $\vec{F}$ is $\text{Div}(\vec{F})$

$$\vec{\nabla} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$$

$$\vec{F} = \langle f, g, h \rangle$$

$$\text{Div}(\vec{F}) = \vec{\nabla} \cdot \vec{F}$$

$$= \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle f, g, h \rangle$$

$$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Example

Find the divergence of $\vec{F} = \langle x^2, y^2, yz \rangle$ at $(1, 0, -1)$

Soln

$$\text{Div}(\vec{F}) = \frac{\partial (x^2)}{\partial x} + \frac{\partial (y^2)}{\partial y} + \frac{\partial (yz)}{\partial z}$$

$$= 2x + 2y$$

$$= 2$$

Curl

It describes the rotation of a vector field in a neighbourhood of P .

NB: ① If $\text{curl}(\vec{F})$ is positive, then counter-clockwise.

② " " " negative, " clockwise

③ " " " zero, " irrotational.

$$\text{Curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & h \end{vmatrix}$$

NB: If $\vec{F}$ is a conservative vector field then $\text{curl} \vec{F} = \vec{0}$

Line Integral Over a Scalar Field

24

The line integral of $f(x,y)$ along C is denoted by

$$\int_C f(x,y) ds$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \|\vec{r}'(t)\|$$

$$\Rightarrow \int_C f(x,y) ds = \int_a^b f(h(t), g(t)) \|\vec{r}'(t)\| dt$$

Example

The temperature of the circular plate $R = \{(x,y) : x^2 + y^2 \leq 1\}$ is $T(x,y) = 100(x^2 + y^2)$. Find the temperature along the path defined by $C : \{(x,y) : x = \cos t, y = \sin t, 0 \leq t \leq 2\pi\}$

$$\int_C T(x,y) ds$$

$$\vec{r}(t) = \langle \cos t, \sin t \rangle$$

$$\vec{r}'(t) = \langle -\sin t, \cos t \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$

$$ds = \|\vec{r}'(t)\| dt$$

$$ds = dt$$

$$\text{Temperature} = \int_0^{2\pi} 100(\cos^2 t + \sin^2 t) dt$$

$$= \int_0^{2\pi} 100 dt$$

$$= 200\pi$$

Example

25

Evaluate $\int_C xy^4 ds$ where C is the right half of the circle $x^2 + y^2 = 16$.

Solution

$$x = 4 \cos t \quad y = 4 \sin t$$

$$-\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \quad (\text{Right half of the circle})$$

$$\frac{dx}{dt} = -4 \sin t \quad \frac{dy}{dt} = 4 \cos t$$

$$ds = \sqrt{(-4 \sin t)^2 + (4 \cos t)^2} dt$$

$$ds = 4 dt$$

$$\int_C xy^4 ds = \int_{-\pi/2}^{\pi/2} (4 \cos t) (4 \sin t)^4 (4) dt$$

$$= \frac{8192}{5}$$

Line Integral Over A Vector Field

$\int_C \vec{F} \cdot d\vec{r}$ This describes the work done in moving a particle over a curve/path C through a force-field or vector field.

$$\text{Work done} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

If F is conservative i.e. $\vec{F} = \nabla f$

$$\Rightarrow \text{Work done} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C \nabla f \cdot d\vec{r}$$

Example

26

Find the work done on a vector field $\vec{F}(x,y,z) = \langle yz^2, xz^2, 2xyz \rangle$ from $(0,0,0)$ to $(1,3,2)$.

Solution

Check whether $\vec{F}$ is conservative. If it is, then find the potential function.

$$\Rightarrow f = xyz^2 + C$$

$$\begin{aligned} \text{Work done} &= (1)(3)(2^2) \\ &= \underline{\underline{12J}} \end{aligned}$$

TRY

Find the work done on the vector field $\vec{F}(x,y) = \langle xe^{2y}, x^2e^{2y} \rangle$ from $A(0,0)$ to $B(-1,1)$.

Answer: $2\pi + 1$

Eigenvalues and Eigenvectors.

27

$$\underset{\text{matrix}}{A} \underset{\text{scalar}}{x} = \lambda \underset{\text{vector}}{x}$$

$$(A - \lambda I) \underline{x} = 0$$

Use $|A - \lambda I| = 0$ --- ① to find the λ 's

Later, substitute λ 's to get $\underline{x}$

Example

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 7 & -4 \\ 5 & -2 \end{pmatrix}$$

Soln

$$(A - \lambda I) \underline{x} = 0$$

$$\begin{pmatrix} 7-\lambda & -4 \\ 5 & -2-\lambda \end{pmatrix} \underline{x} = 0$$

$$\begin{vmatrix} 7-\lambda & -4 \\ 5 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\lambda = 2, \lambda = 3$$

when $\lambda = 2$

$$\begin{pmatrix} 5 & -4 \\ 5 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad R_1 - R_2 \rightarrow R_2$$

$$\begin{pmatrix} 5 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

x_2 is free

$$5x_1 - 4x_2 = 0$$

$$x_1 = \frac{4}{5}x_2$$

$$\{ \underline{x} : \underline{x} = \alpha \begin{pmatrix} 4/5 \\ 1 \end{pmatrix} \}$$

when $\lambda = 3$

$$\{ \underline{x} : \underline{x} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \}$$

$$\text{let } x_2 = \alpha$$

$$\Rightarrow x_1 = \frac{4}{5}\alpha$$

$\therefore$ the eigenvalues are $\lambda = 2, \lambda = 3$ and

the eigenvectors are $\begin{pmatrix} 4/5 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

TRY

$$\textcircled{1} \begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix}$$

$$\textcircled{2} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\textcircled{3} \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

$$\textcircled{4} \begin{pmatrix} 1 & -4 & -4 \\ 8 & -11 & -8 \\ -8 & 8 & 5 \end{pmatrix}$$

Note

- ① The eigenvalues of A^2 and A^{-1} are λ^2 and λ^{-1} respectively.
- ② The sum of the eigenvalues of A is equal to the trace of A .
- ③ The product of the eigenvalues of A is equal to $\det(A)$.

Similarity Transformation and Diagonalization

Suppose the $n \times n$ matrix A has n linearly independent eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$. Put them into columns of an eigenvector matrix S . Then S^{-1} is the eigenvalue matrix Λ .

$$\Lambda = S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \ddots & \lambda_n \end{bmatrix}$$

The matrix A is diagonalized.

$$\Lambda = S^{-1}AS$$

$$S\Lambda S^{-1} = S S^{-1} A S S^{-1}$$

$$S\Lambda S^{-1} = A$$

$$A^2 = S\Lambda^2 S^{-1}$$

$$A^k = S\Lambda^k S^{-1}$$

Note: * A^k has the same S and the eigenvalue matrix of A^k is Λ^k for $k \geq 2$.

* A is diagonalizable if every eigenvalue has enough eigenvectors.

Theorem: An $n \times n$ matrix with n distinct eigenvalues is diagonalized.